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Error estimates of a spectral Petrov–Galerkin method for two-sided fractional reaction–diffusion equations

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A B S T R A C T

We study regularity and the spectral method for two-sided fractional diffusion equations with a reaction term. We show that the regularity of the solution in weighted Sobolev spaces can be greatly improved compared to that in standard Sobolev spaces. With this regularity, we prove an optimal error estimate for the spectral Petrov–Galerkin method. Numerical results are presented to verify our theoretical convergence orders.

1. Introduction

Anomalous diffusion has been widely used to investigate transport dynamics in complex systems, such as underground environmental problem [16], fluid flow in porous materials [4], anomalous transport in biology [17], etc. Many mathematical models are developed to study anomalous diffusion. Some of these models are based on a linear equation for diffusion on fractals [31], a linear differential Fisher's information theory [37] and Levy description of anomalous diffusion in dynamical systems [20]. In particular, fractional differential equations (FDEs) can serve as an accurate model of the anomalous diffusion, e.g. super-diffusion process in [30].

Due to the extraordinary capabilities in modeling, numerical simulation and analysis of the fractional differential equations have attracted much attentions and growing interest. Extensive numerical methods have been investigated in recent decades e.g. finite difference methods [11,12,24,29,36,39], finite element methods [8,9,13,19,40–42,49,50], spectral methods [5,9,18,23,27,43–45,48], discontinuous Galerkin methods [38], finite volume methods [34], etc.

Despite the significant number of numerical methods for FDEs, regularity of solutions to FDEs is not thoroughly investigated, especially the regularity well suited for error analysis. In literature, it is assumed that solutions are sufficiently smooth. However, it has been pointed out in [5,9,19,21,22,35,40] that the regularity of solutions to FDEs can be very low. Recently there have been works on discussing the low regularity of solutions; see e.g. [9,15,46].

In this paper, we consider the following two-sided fractional diffusion equation with a reaction term

$$\mathcal{L}_\theta^\alpha u + \mu u = f(x), \quad x \in \Omega = (a, b), \tag{1.1}$$

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with homogeneous boundary conditions

$$u(a) = u(b) = 0, \tag{1.2}$$

and a nonnegative reaction coefficient μ, a given function $f(x)$, $\mathcal{L}_\theta^\alpha := -[\theta\, {}_aD_x^\alpha + (1-\theta)\, {}_xD_b^\alpha]$ with $\theta \in [0, 1]$ and $\alpha \in (1, 2)$. Here ${}_aD_x^\alpha$ and ${}_xD_b^\alpha$ are left- and right-sided Riemann–Liouville operators, defined as follows (see e.g. [32,33])

$$ {}_aD_x^\alpha u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_a^x \frac{u(\xi)}{(x-\xi)^{\alpha-1}} d\xi, \quad x > a, \tag{1.3}$$

and

$$ {}_xD_b^\alpha u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_x^b \frac{u(\xi)}{(\xi-x)^{\alpha-1}} d\xi, \quad x < b. \tag{1.4}$$

The model equations can be obtained by incorporating a fractional Fick's law into a conventional local mass balance law [6,47]. Here θ is a skewness parameter; see [4,7]. When $\theta = 1/2$ it becomes the symmetrical diffusion equation which is closely related to the fractional Laplacian. When $\theta = 1$ or 0 it reduces to the extreme of asymmetrical case which has been mostly studied among the existing literature.

Ervin and Roop [8] established uniqueness and existence of the solution to (1.1). Jin et al [19] showed the regularity of the solution in extremely asymmetrical case $\theta = 1$ or 0 in (1.1). The starting point of their approach is to seek the closed form of strong solution in one-sided case by acting on the fractional integral operator on the both side of the equation. The similar idea occurs in the paper [21,22] to find the dominant singularity of leading term in the solution. However the approach in above paper does not work for the general two-sided case since there is no explicit inverse operator for two-sided fractional derivatives. The first attempt to discuss the regularity of the equation in two-sided case is from the work by Ervin et al in [9], where they conjectured the regularity of solution in standard Sobolev space by seeking a closed form expression for the kernel of fractional diffusion operator for $\mu = 0$ and assuming the data f sufficiently smooth in (1.1). A solution is numerically shown to be in the standard Sobolev space H^γ with regularity index $\gamma = \min\{\sigma, \sigma^*\} + 1/2 - \epsilon$, where $\epsilon > 0$ is an arbitrary small number, σ and σ^* are constants (see Lemma 2.1) depending on the order α and parameter θ in (1.1). In particular the regularity index is $\alpha/2 + 1/2 - \epsilon$ for the symmetrical case $\theta = 1/2$ and $\alpha - 1/2 + \epsilon$ for the one-sided asymmetrical case $\theta = 1$ or 0.

This work is devoted to the study of regularity and verification of this analysis using the spectral method for (1.1) when $\mu \neq 0$. From the weakly singular kernel of fractional derivatives, solutions of fractional differential equations naturally inherit weak singularity. According to [9], the solution to (1.1) when $\mu = 0$ can be written as the product, i.e., $u = (b-x)^\sigma (x-a)^{\sigma^*} \tilde{u}$, where $\tilde{u}$ is a smooth function depending on the smoothness of the right hand side f and can be much smoother than u. Based on the relation in Lemma 2.1, we expand the function $\tilde{u}$ using Jacobi polynomials and apply a Fourier-type analysis of $\tilde{u}$. For $\tilde{u}$, we show that the solution has a regularity index $2\alpha + 1 - \epsilon$ in non-uniformly weighted Sobolev spaces and is higher than that in the usual Sobolev spaces. This work is a continuation of our recent work [15,46] which gave the regularity for the fractional diffusion equation with fractional Laplacian by Fourier analysis and bootstrapping technique for enhancing the regularity.

Compared to other numerical methods such as finite difference/finite volume/finite element methods, the advantage of spectral methods is their spectral accuracy which means their convergence orders can change with regularity of solutions. This advantage makes spectral methods one of the best tools to test regularity of solutions in practice. In this work, we present a spectral Petrov–Galerkin method and prove the optimal convergence of such a method. In the symmetrical case $\theta = 1/2$, the spectral Petrov–Galerkin method reduces to spectral Galerkin method and an optimal error estimate is obtained in [15]. In this work, we focus on the non-symmetrical case $\theta \neq 1/2$. Due to the asymmetry of the fractional operators, the analysis is more challenging than that in [15].

The contribution of this work is summarized as follows.

- We present the regularity in terms of the right hand side function f for the two-sided equation with a reaction term. Here we discuss of a large class of right hand side functions f. Here f can be smooth enough or only in L^2.
- We show the higher regularity of $\tilde{u}$ in the weighted Sobolev spaces than the regularity of the solution u in usual Sobolev spaces. The benefit of using weighted Sobolev spaces is that these spaces can better accommodate boundary singularity.
- We present the spectral Petrov–Galerkin method to verify the analyzed regularity and prove its optimal error estimate; see Theorem 4.2.

The rest of this paper is arranged as follows. In Section 2, we introduce some basic notations and recall some properties of Jacobi polynomials and non-uniformly weighted Sobolev spaces. Some lengthy but important auxiliary materials are presented in Appendix. In Section 3, we present the regularity of the two-sided fractional diffusion equations using Fourier type analysis and a bootstrapping technique. In Section 4, we present a Petrov–Galerkin method and provide error estimates as well. Several numerical results are shown to verify the theoretical convergence order in Section 5. Finally, we make some concluding remarks.

2. Preliminary

In this section, we introduce Jacobi polynomials, Jacobi-weighted Sobolev spaces and basic facts for fractional derivatives we will use.

We consider the interval $\Omega = (-1, 1)$ for simplicity. Denote by $L^2_{\omega^{\gamma,\beta}}(\Omega)$ the space with the inner product and norm defined by

$$(u, v)_{\omega^{\gamma,\beta},\Omega} = \int_{\Omega} uv\omega^{\gamma,\beta} dx, \quad \|u\|_{\omega^{\gamma,\beta},\Omega} = (u, u)^{\frac{1}{2}}_{\omega^{\gamma,\beta},\Omega} \tag{2.1}$$

where $\omega^{\gamma,\beta} = (1-x)^{\gamma}(1+x)^{\beta}$, $\gamma, \beta > -1$. When $\gamma = \beta = 0$, we will drop ω from the above notations. We also drop the domain Ω from the notation for simplicity without incurring confusion.

The Jacobi polynomials $P_n^{\gamma,\beta}(x)$ are mutually orthogonal: for $\gamma,\ \beta > -1$,

$$\int_{-1}^{1} (1-x)^{\gamma}(1+x)^{\beta} P_m^{\gamma,\beta}(x) P_n^{\gamma,\beta}(x)\, dx = h_n^{\gamma,\beta} \delta_{nm}. \tag{2.2}$$

Here δ_{nm} is equal to 1 if $n = m$ and zero otherwise, and

$$h_n^{\gamma,\beta} = \left\| P_n^{\gamma,\beta} \right\|^2_{\omega^{\gamma,\beta}} = \frac{2^{\gamma+\beta+1}}{2n+\gamma+\beta+1} \times \frac{\Gamma(n+\gamma+1)\Gamma(n+\beta+1)}{\Gamma(n+\gamma+\beta+1)\Gamma(n+1)}. \tag{2.3}$$

The following asymptotic formula for a ratio of two gamma functions holds

$$\lim_{n\to\infty} \frac{\Gamma(n+\delta)}{n^{\delta-\gamma}\Gamma(n+\gamma)} = \lim_{n\to\infty} \left[1 + \frac{(\delta-\gamma)(\delta+\gamma-1)}{2n} + \mathcal{O}(n^{-2}) \right] = 1. \tag{2.4}$$

We say that a_n is equivalent to b_n if there exits c_1 and c_2 such that $c_1 a_n \le b_n \le c_2 a_n$ asymptotically and denote them by $a_n \approx b_n$. By (2.4), we know that $h_n^{\gamma,\beta} \approx \frac{1}{2n+\gamma+\beta+1}$.

To incorporate singularities at the endpoints, we introduce the following non-uniformly Jacobi-weighted Sobolev space (see e.g. [3,10]),

$$B^m_{\omega^{\gamma,\beta}} := \left\{ u \mid \partial_x^k u \in L^2_{\omega^{\gamma+k,\beta+k}},\ k = 0, 1, \dots, m \right\},\ m \text{ is a nonnegative integer}, \tag{2.5}$$

which is equipped with the following norm

$$\|u\|_{B^m_{\omega^{\gamma,\beta}}} = \left(\sum_{k=0}^{m} |u|^2_{B^k_{\omega^{\gamma,\beta}}} \right)^{1/2}, \quad |u|_{B^k_{\omega^{\gamma,\beta}}} = \left\| \partial_x^k u \right\|_{\omega^{\gamma+k,\beta+k}}. \tag{2.6}$$

When m is not an integer, the space can be defined via classical interpolation method, e.g. K- method; see [1]. For functions in $B^s_{\omega^{\gamma,\beta}}$, we may introduce the following norm:

$$\| |u| \|^2_{B^s_{\omega^{\gamma,\beta}}} = \sum_{n=0}^{\infty} (u_n^{\gamma,\beta})^2 \frac{1+n^{2s}}{2n+\gamma+\beta+1}, \tag{2.7}$$

where $u_n^{\gamma,\beta}$ are the coefficients of the Jacobi-Fourier expansion of u in terms of $P_n^{\gamma,\beta}$. When $s = m$ is an integer number, the norm (2.7) is equivalent to the norm (2.6); see [14].

The following *pseudo-eigen functions* for the fractional diffusion operator in [9] are essential to analyze the regularity.

Lemma 2.1 [9]**.** *For the nth order Jacobi polynomial $P_n^{\sigma,\sigma^*}(x)$, it holds that*

$$\mathcal{L}_\theta^\alpha [\omega^{\sigma,\sigma^*} P_n^{\sigma,\sigma^*}(x)] = \lambda_{\theta,n}^\alpha P_n^{\sigma^*,\sigma}(x), \tag{2.8}$$

where

$$\lambda_{\theta,n}^\alpha = -\frac{\sin(\pi\alpha)}{\sin(\pi\sigma)+\sin(\pi\sigma^*)} \times \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)},$$

and $\sigma^* = \alpha - \sigma$ and σ is determined by the following equation:

$$\theta = \frac{\sin(\pi\sigma^*)}{\sin(\pi\sigma^*)+\sin(\pi\sigma)}. \tag{2.9}$$

Remark 2.1. To ensure that (2.9) is uniquely solvable, we restrict σ and σ^* into the interval (0,1]. In particular, $\sigma = 1$ and $\sigma^* = \alpha - 1$ for $\theta = 1; \sigma = \alpha - 1$ and $\sigma^* = 1$ for $\theta = 0$, and $\lambda_{0,n}^\alpha = \lambda_{1,n}^\alpha = \frac{\Gamma(\alpha+n+1)}{n!}$. These constants can be uniquely found using a Newton's method. See Section 5 for details and examples of these constants.

Throughout the paper, C or c denote generic constants and are independent of any functions and of the truncation parameter N.

3. Regularity

In this section, we present our regularity results in weighted Sobolev spaces and their proofs, see Theorems 3.1 and 3.2. Some of the details of proofs which are less relevant are presented in Appendix.

The weak formulation of the problem (1.1) and (1.2) is to find $u \in H_0^{\alpha/2}$, such that

$$(\mathcal{L}_\theta^\alpha u, v) + \mu(u, v) = (f, v), \quad \forall v \in H_0^{\alpha/2}. \tag{3.1}$$

The wellposedness of the problem (1.1) and (1.2) has been established in [8], which is stated as follows.

Lemma 3.1 [8]**.** *For the problem (1.1) and (1.2), there exits a unique solution $u \in H_0^{\alpha/2}$ such that $\|u\|_{H^{\alpha/2}} \le \|f\|_{H^{-\alpha/2}}$, where $H^{-\alpha/2}$ is the dual space of $H_0^{\alpha/2}$ with respect to the inner product in L^2 space.*

Here we require that $f \in L^2_{\omega^{\sigma^*,\sigma}}$ and thus $u \in L^\infty$. In fact, we have by (A.4) that

$$\|f\|_{H^{-\alpha/2}} = \sup_{v\in H_0^{\alpha/2},\, v\neq 0} \frac{(f,v)}{\|v\|_{H^{\alpha/2}}} \le \sup_{v\in H_0^{\alpha/2},\, v\neq 0} \frac{\|f\|_{\omega^{\sigma^*,\sigma}}\|v\|_{\omega^{-\sigma^*,-\sigma}}}{\|v\|_{H^{\alpha/2}}} \le C\|f\|_{\omega^{\sigma^*,\sigma}}.$$

This implies $f \in H^{-\alpha/2}$ and thus by Lemma 3.1 there exists a unique solution u which belongs to $H_0^{\alpha/2}$. By the Sobolev embedding inequality, $u \in L^\infty$.

We are now at the position to present the regularity of the two-sided FDE (1.1). In Theorem 3.1, we first consider the regularity of the solution under the condition that forcing term f belongs to weighted Sobolev space $B^r_{\omega^{\sigma^*,\sigma}}$, where the index σ^*, σ is defined in Lemma 2.1. Next, in Theorem 3.2, we study more stronger condition that f belongs to weighted Sobolev space in terms of Jacobi index $\sigma^*-1, \sigma-1$ instead of σ^*, σ, and further improve the regularity of solution.

Theorem 3.1 (Regularity in weighted Sobolev spaces, I)**.** *Assume that $f \in B^r_{\omega^{\sigma^*,\sigma}}$ with $r \ge 0$. If $\mu = 0$, then $\omega^{-\sigma,-\sigma^*}u \in B^{\alpha+r}_{\omega^{\sigma,\sigma^*}}$. If $\mu > 0$, then $\omega^{-\sigma,-\sigma^*}u \in B^{\alpha\wedge r+\alpha-\epsilon}_{\omega^{\sigma,\sigma^*}}$ with $\epsilon > 0$. Here $\alpha \wedge r = \min\{\alpha, r\}$.*

Proof. For $\mu = 0$, we write $u = \omega^{\sigma,\sigma^*}\sum_{n=0}^\infty u_n P_n^{\sigma,\sigma^*}(x)$. Then with Lemma 2.1 we have $u_n = (\lambda^\alpha_{\theta,n})^{-1} f_n$ from the equation $\mathcal{L}_\theta^\alpha u = f$ and $f = \sum_{n=0}^\infty f_n P_n^{\sigma^*,\sigma}(x)$. Since $f \in B^r_{\omega^{\sigma^*,\sigma}}$, by the definition (2.7), we have

$$\||f|\|^2_{B^r_{\omega^{\sigma^*,\sigma}}} = \sum_{n=0}^\infty \frac{f_n^2(1+n^{2r})}{2n+\alpha+1} < \infty.$$

By (2.4), we know that $\lambda^\alpha_{\theta,n} \approx n^\alpha$. It follows that

$$\begin{aligned}\||\omega^{-\sigma,-\sigma^*}u|\|^2_{B^{\alpha+r}_{\omega^{\sigma,\sigma^*}}} &= \sum_{n=0}^\infty \frac{u_n^2(1+n^{2(\alpha+r)})}{2n+\alpha+1} \\ &= \sum_{n=0}^\infty \frac{f_n^2(\lambda^\alpha_{\theta,n})^{-2}(1+n^{2(\alpha+r)})}{2n+\alpha+1} \\ &\approx \sum_{n=0}^\infty \frac{f_n^2 n^{-2\alpha}(1+n^{2(\alpha+r)})}{2n+\alpha+1} < \infty,\end{aligned}$$

which implies that $\omega^{-\sigma,-\sigma^*}u \in B^{\alpha+r}_{\omega^{\sigma,\sigma^*}}$.

Now consider $\mu > 0$. We use a bootstrapping technique to obtain higher regularity. First, we can obtain that $u = \omega^{\sigma,\sigma^*}\tilde{u}$ and $\tilde{u} \in B^\alpha_{\omega^{\sigma,\sigma^*}}$. In fact, we have from $u \in L^\infty$ that $\tilde{f} = f - \mu u \in B^0_{\omega^{\sigma^*,\sigma}}$. From the equation $\mathcal{L}_\theta^\alpha u = \tilde{f}$ and using the conclusion when $\mu = 0$, we have $\tilde{u} = \omega^{-\sigma,-\sigma^*}u \in B^\alpha_{\omega^{\sigma,\sigma^*}}$. Then $\tilde{u} = \omega^{-\sigma,-\sigma^*}u \in B^1_{\omega^{\sigma,\sigma^*}}$ and using Lemma A.5 leads to $u \in B^1_{\omega^{\sigma^*,\sigma}}$.

Furthermore, we obtain that $\tilde{f} = f - \mu u \in B^{r\wedge 1}_{\omega^{\sigma^*,\sigma}}$. Using the conclusion above again, we have $\tilde{u} \in B^{r\wedge 1+\alpha}_{\omega^{\sigma,\sigma^*}}$. If $r \le 1$, then we get the conclusion. When $r > 1$, $\tilde{u} \in B^{r+\alpha}_{\omega^{\sigma,\sigma^*}}$ and hence $\tilde{u} \in B^2_{\omega^{\sigma,\sigma^*}}$, which, by Lemma A.6, we have $u \in B^{\alpha-\epsilon}_{\omega^{\sigma^*,\sigma}}$ with $\epsilon > 0$ arbitrary.

Finally, by the fact that $\tilde{f} = f - \mu u \in B^{(\alpha-\epsilon)\wedge r}_{\omega^{\sigma^*,\sigma}}$ and the conclusion for $\mu = 0$, we have $\omega^{-\sigma,-\sigma^*}u \in B^{\alpha\wedge r+\alpha-\epsilon}_{\omega^{\sigma,\sigma^*}}$. □

Theorem 3.2 (Regularity in weighted Sobolev spaces, II)**.** *Assume that $f \in B^r_{\omega^{\sigma^*-1,\sigma-1}}$, with $r \ge 0$. If $\mu = 0$, then $\omega^{-\sigma,-\sigma^*}u \in B^{\alpha+r-\epsilon}_{\omega^{\sigma-1,\sigma^*-1}}$ with $\epsilon > 0$. If $\mu > 0$, $\omega^{-\sigma,-\sigma^*}u \in B^{(\alpha+1)\wedge r+\alpha-\epsilon}_{\omega^{\sigma,\sigma^*}}$ and $\mathcal{L}_\theta^\alpha u \in B^{r\wedge(\alpha+1)-\epsilon}_{\omega^{\sigma^*,\sigma}}$.*

Proof. Since $f \in B^r_{\omega^{\sigma^*-1,\sigma-1}}$, then $f \in L^2$. Following the last theorem, we know that there exists a unique solution $u \in H_0^{\alpha/2}$. Consider first $\mu = 0$. From Corollary A.1 in Appendix, we have the following relations: for $n \ge 0$

$$P_n^{\sigma-1,\sigma^*-1} = A_n^{\sigma-1,\sigma^*-1}P_{n-2}^{\sigma,\sigma^*} + B_n^{\sigma-1,\sigma^*-1}P_{n-1}^{\sigma,\sigma^*} + C_n^{\sigma-1,\sigma^*-1}P_n^{\sigma,\sigma^*}, \tag{3.2}$$

$$P_n^{\sigma^*-1,\sigma-1} = A_n^{\sigma-1,\sigma^*-1}P_{n-2}^{\sigma^*,\sigma} - B_n^{\sigma-1,\sigma^*-1}P_{n-1}^{\sigma^*,\sigma} + C_n^{\sigma-1,\sigma^*-1}P_n^{\sigma^*,\sigma}, \tag{3.3}$$

where $A_n^{\sigma-1,\sigma^*-1}$, $B_n^{\sigma-1,\sigma^*-1}$ and $C_n^{\sigma-1,\sigma^*-1}$ are defined in Corollary A.1 and $P_{-2}^{\gamma,\beta} \equiv P_{-1}^{\gamma,\beta} \equiv 0$. Throughout the proof, to simplify the notations, we drop the superscript $\sigma-1, \sigma^*-1$ for A_n, B_n and C_n and abbreviate $\lambda_{\theta,n}^{\alpha}$ as λ_n. From (3.2), we have

$$\sum_{n=0}^{\infty} u_n P_n^{\sigma-1,\sigma^*-1} = \sum_{n=0}^{\infty} u_n (A_n P_{n-2}^{\sigma,\sigma^*} + B_n P_{n-1}^{\sigma,\sigma^*} + C_n P_n^{\sigma,\sigma^*})$$
$$= \sum_{n=0}^{\infty} (u_{n+2}A_{n+2} + u_{n+1}B_{n+1} + u_n C_n) P_n^{\sigma,\sigma^*}. \tag{3.4}$$

It follows from Lemma 2.1 that

$$\mathcal{L}_\theta^\alpha u = \mathcal{L}_\theta^\alpha \left(\omega^{\sigma,\sigma^*} \sum_{n=0}^{\infty} u_n P_n^{\sigma-1,\sigma^*-1} \right)$$
$$= \mathcal{L}_\theta^\alpha \left[\omega^{\sigma,\sigma^*} \sum_{n=0}^{\infty} (u_{n+2}A_{n+2} + u_{n+1}B_{n+1} + u_n C_n) P_n^{\sigma,\sigma^*} \right]$$
$$= \sum_{n=0}^{\infty} \lambda_n (u_{n+2}A_{n+2} + u_{n+1}B_{n+1} + u_n C_n) P_n^{\sigma^*,\sigma}. \tag{3.5}$$

From (3.3), we have

$$\sum_{n=0}^{\infty} f_n P_n^{\sigma^*-1,\sigma-1} = \sum_{n=0}^{\infty} f_n (A_n P_{n-2}^{\sigma^*,\sigma} - B_n P_{n-1}^{\sigma^*,\sigma} + C_n P_n^{\sigma^*,\sigma})$$
$$= \sum_{n=0}^{\infty} (f_{n+2}A_{n+2} - f_{n+1}B_{n+1} + f_n C_n) P_n^{\sigma^*,\sigma}. \tag{3.6}$$

Substituting (3.5) and (3.6) into (1.1) leads to

$$\sum_{n=0}^{\infty} \lambda_n (u_{n+2}A_{n+2} + u_{n+1}B_{n+1} + u_n C_n) P_n^{\sigma^*,\sigma} = \sum_{n=0}^{\infty} (f_{n+2}A_{n+2} - f_{n+1}B_{n+1} + f_n C_n) P_n^{\sigma^*,\sigma}. \tag{3.7}$$

Taking the inner product of $P_n^{\sigma^*,\sigma}$ with respect to weigh function $\omega^{\sigma^*,\sigma}$ over both sides of the last Eq. (3.7) and by the orthogonality of Jacobi polynomials, we arrive at

$$u_{n+2}A_{n+2} + u_{n+1}B_{n+1} + u_n C_n = \frac{1}{\lambda_n}(f_{n+2}A_{n+2} - f_{n+1}B_{n+1} + f_n C_n)$$
$$= F_{n+2}A_{n+2} + F_{n+1}B_{n+1} + F_n C_n + \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+2}}\right) f_{n+2}A_{n+2} - \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{n+1}}\right) f_{n+1}B_{n+1}, \tag{3.8}$$

where $F_n = f_n/\lambda_n$. Multiplying $P_n^{\sigma-1,\sigma^*-1}$ on both sides and summing over n, using (3.4) again, we obtain

$$\sum_{n=0}^{\infty} u_n P_n^{\sigma-1,\sigma^*-1} = \sum_{n=0}^{\infty} F_n P_n^{\sigma-1,\sigma^*-1} + \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+2}}\right) f_{n+2}A_{n+2} P_n^{\sigma,\sigma^*} - \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{n+1}}\right) f_{n+1}B_{n+1} P_n^{\sigma,\sigma^*}. \tag{3.9}$$

Replacing n by k in the above equation and taking the product of $P_n^{\sigma-1,\sigma^*-1}$ with respect to the weight function $\omega^{\sigma-1,\sigma^*-1}$ leads to

$$u_n = F_n + \sum_{k=n}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+2}}\right) f_{k+2}A_{k+2} (h_n^{\sigma-1,\sigma^*-1})^{-1} (P_k^{\sigma,\sigma^*}, P_n^{\sigma-1,\sigma^*-1})_{\omega^{\sigma-1,\sigma^*-1}}$$
$$- \sum_{k=n}^{\infty} \left(\frac{1}{\lambda_k} + \frac{1}{\lambda_{k+1}}\right) f_{k+1}B_{k+1} (h_n^{\sigma-1,\sigma^*-1})^{-1} (P_k^{\sigma,\sigma^*}, P_n^{\sigma-1,\sigma^*-1})_{\omega^{\sigma-1,\sigma^*-1}},$$

where $h_n^{\sigma-1,\sigma^*-1}$ is defined in (2.3). Notice that $|A_k| \le C$ and $|B_k| \le C/k$ in Corollary A.1. By Lemma A.4, we have

$$|u_n| \le \frac{|f_n|}{\lambda_n} + C\sum_{k=n}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+2}}\right) |f_{k+2}| + C\sum_{k=n}^{\infty} \left(\frac{1}{\lambda_k} + \frac{1}{\lambda_{k+1}}\right) \frac{1}{k} |f_{k+1}|. \tag{3.10}$$

For the second term on the right hand side, we have

$$C\sum_{k=n}^{\infty}\left(\frac{1}{\lambda_k}-\frac{1}{\lambda_{k+2}}\right)|f_{k+2}| \le C\left(\sum_{k=n}^{\infty}|f_k|^2k^{2r-1}\right)^{\frac{1}{2}}\left(\sum_{k=n}^{\infty}\frac{1}{k^{2\alpha+2r+1}}\right)^{\frac{1}{2}}$$
$$\le C\left(\sum_{k=n}^{\infty}|f_k|^2k^{2r-1}\right)^{\frac{1}{2}}\left(\int_n^{\infty}\frac{1}{x^{2\alpha+2r+1}}dx\right)^{\frac{1}{2}}$$
$$\le C\left(\sum_{k=n}^{\infty}|f_k|^2k^{2r-1}\right)^{\frac{1}{2}}n^{-\alpha-r} \le C\|f\|_{B^r_{\omega^{\sigma^*,\sigma}}}n^{-\alpha-r}.$$

The last term on the right hand side can be treated similarly.

Therefore, we have

$$|u_n| \le C|f_n|n^{-\alpha}+Cn^{-\alpha-r}. \tag{3.11}$$

By Definition (2.7), we have $\omega^{-\sigma,-\sigma^*}u \in B^{r+\alpha-\epsilon}_{\omega^{\sigma-1,\sigma^*-1}}$ with $\epsilon>0$ arbitrary.

When $\mu>0$, we apply the bootstrapping technique. By $\mathcal{L}^{\alpha}_{\theta}u = f-\mu u \in L^2_{\omega^{\sigma-1,\sigma^*-1}}$ and the conclusion above, we have $\omega^{-\sigma,-\sigma^*}u \in B^{\alpha-\epsilon}_{\omega^{\sigma-1,\sigma^*-1}}$ and hence $\omega^{-\sigma,-\sigma^*}u \in B^{1}_{\omega^{\sigma-1,\sigma^*-1}}$, which leads to $u \in B^{1}_{\omega^{\sigma^*-1,\sigma-1}}$. Thus $\mathcal{L}^{\alpha}_{\theta}u = f-\mu u \in B^{r\wedge 1}_{\omega^{\sigma^*-1,\sigma-1}}$ and by the conclusion for $\mu=0$, we have $\omega^{-\sigma,-\sigma^*}u \in B^{r\wedge 1+\alpha-\epsilon}_{\omega^{\sigma-1,\sigma^*-1}}$.

If $r\le 1$, we have reached the conclusion since $B^{r\wedge 1+\alpha-\epsilon}_{\omega^{\sigma-1,\sigma^*-1}} \subseteq B^{r\wedge 1+\alpha-\epsilon}_{\omega^{\sigma,\sigma^*}}$. If $r>1$, then $\omega^{-\sigma,-\sigma^*}u \in B^{2}_{\omega^{\sigma-1,\sigma^*-1}}$, by Lemma A.5 which further leads to $u \in B^{2}_{\omega^{\sigma^*-1,\sigma-1}}$. Therefore, we have $\mathcal{L}^{\alpha}_{\theta}u = f-\mu u \in B^{r\wedge 2}_{\omega^{\sigma^*-1,\sigma-1}}$. Then by the conclusion for $\mu=0$, we have $\omega^{-\sigma,-\sigma^*}u \in B^{r\wedge 2+\alpha-\epsilon}_{\omega^{\sigma-1,\sigma^*-1}}$. If $r\le 2$, we have the conclusion since $B^{r\wedge 2+\alpha-\epsilon}_{\omega^{\sigma-1,\sigma^*-1}} \subseteq B^{r\wedge 2+\alpha-\epsilon}_{\omega^{\sigma,\sigma^*}}$. If $r>2$, we have $\omega^{-\sigma,-\sigma^*}u \in B^{3}_{\omega^{\sigma-1,\sigma^*-1}}$, and by Lemma A.7, $u \in B^{\alpha+1-\epsilon}_{\omega^{\sigma^*,\sigma}}$. Thus $\mathcal{L}^{\alpha}_{\theta}u = f-\mu u \in B^{r\wedge(\alpha+1-\epsilon)}_{\omega^{\sigma^*,\sigma}}$ and by the conclusion for $\mu=0$, we have $\omega^{-\sigma,-\sigma^*}u \in B^{r\wedge(\alpha+1)+\alpha-\epsilon}_{\omega^{\sigma,\sigma^*}}$. □

Remark 3.1. Throughout the paper we consider the constant μ for simplicity and it is straightforward to extend our results to variable coefficients cases, for example, $\mu(x) \in C^3$.

4. Spectral Petrov–Galerkin method

In this section, we consider a spectral Petrov–Galerkin method and present an optimal error estimate and its proof based on the obtained regularity in Section 3.

The spectral Petrov–Galerkin method is to find $u_N \in U_N$ such that

$$(\mathcal{L}^{\alpha}_{\theta}u_N, v_N)+\mu(u_N,v_N) = (f,v_N), \quad \forall v_N \in V_N. \tag{4.1}$$

The method is implicitly discussed in [9] and it is fully discussed in [28] when $\mu=0$. Here we define the finite dimensional spaces $U_N := \omega^{\sigma,\sigma^*}\mathbb{P}_N = \mathrm{Span}\{\phi_0,\phi_1,\dots,\phi_N\}$ and $\phi_k(x) := (1-x)^{\sigma}(1+x)^{\sigma^*}P^{\sigma,\sigma^*}_k(x)$, $V_N := \omega^{\sigma^*,\sigma}\mathbb{P}_N = \mathrm{Span}\{\varphi_0,\varphi_1,\dots,\varphi_N\}$ and $\varphi_k(x) := (1-x)^{\sigma^*}(1+x)^{\sigma}P^{\sigma^*,\sigma}_k(x)$.

4.1. Implementation

In this subsection, we first describe the implementation for the spectral Petrov–Galerkin method.

For implementation, plugging $u_N=\sum_{n=0}^{N}\hat{u}_n\phi_n(x)$ in (4.1) and taking $v_N=\varphi_k(x)$, we obtain from Lemma 2.1 and the orthogonality of Jacobi polynomials that

$$\lambda^{\alpha}_{\theta,k}h^{\sigma^*,\sigma}_k\hat{u}_k+\mu\sum_{n=0}^{N}M_{k,n}\hat{u}_n = (f,\varphi_k), \quad k=0,1,2,\dots,N, \tag{4.2}$$

where $\lambda^{\alpha}_{\theta,k}$ is defined in Lemma 2.1 and

$$M_{k,n} = \int_{-1}^{1}(1-x^2)^{\alpha}P^{\sigma,\sigma^*}_n(x)P^{\sigma^*,\sigma}_k(x)\,dx. \tag{4.3}$$

To $M_{k,n}$, we apply Gauss–Jacobi quadrature rule, e.g.

$$M_{k,n} \approx \sum_{j=0}^{N}P^{\sigma,\sigma^*}_n(x_j)P^{\sigma^*,\sigma}_k(x_j)w_j.$$

Here x_j's are the zeros of Jacobi polynomial $P^{\alpha,\alpha}_{N+1}(x)$, w_j's are the corresponding quadrature weights. The quadrature rule here is exact since $n+k\le 2N$ while the quadrature rule is exact for all $(2N+1)$th order polynomials. The integral in $S_{n,k}$ can be calculated similarly. To find $f_k=(f,\varphi_k)$, we use a different Gauss–Jacobi quadrature rule: $f_k \approx \tilde{f}_k = \sum_{j=0}^{\mathsf{N}}f(\mathsf{x}_j)P^{\sigma^*,\sigma}_k(\mathsf{x}_j)\mathsf{w}_j$. Here x_j's are the roots of Jacobi polynomial $P^{\sigma^*,\sigma}_{\mathsf{N}+1}(x)$, w_j's are the corresponding quadrature weights. Here N is taken to be large enough so that the quadrature errors can be ignored.

4.2. Error analysis of the spectral Petrov–Galerkin method

The well-posdeness of the discrete problem (4.2) can be readily shown by the Lax-Milgram theorem.

Before presenting the error estimates, we need the following approximation properties. Define the $L^2_{\omega^{\sigma,\sigma^*}}$-orthogonal projection $\mathcal{P}_N^{\sigma,\sigma^*}: L^2_{\omega^{\sigma,\sigma^*}} \to \mathbb{P}_N$ such that $(\mathcal{P}_N^{\sigma,\sigma^*}u - u, v)_{\omega^{\sigma,\sigma^*}} = 0$ for any $v \in \mathbb{P}_N$, or equivalently $\mathcal{P}_N^{\sigma,\sigma^*}(u)(x) = \sum_{n=0}^{N} \hat{u}_n P_n^{\sigma,\sigma^*}(x)$. Denote $\Pi_N^{\sigma,\sigma^*}u := \omega^{\sigma,\sigma^*}\mathcal{P}_N^{\sigma,\sigma^*}(\omega^{-\sigma,-\sigma^*}u)$.

Lemma 4.1 (Estimate of projection error)**.** *Let $\omega^{-\sigma,-\sigma^*}u \in B^m_{\omega^{\sigma,\sigma^*}}$ and $\mathcal{L}_\theta^\alpha u \in B^m_{\omega^{\sigma^*,\sigma}}$. Then, for $0 \le m \le N$, we have the following estimates:*

$$\|u - \Pi_N^{\sigma,\sigma^*}u\|_{\omega^{-\sigma,-\sigma^*}} \le cN^{-m}|\omega^{-\sigma,-\sigma^*}u|_{B^m_{\omega^{\sigma,\sigma^*}}}, \tag{4.4}$$

and

$$\|\mathcal{L}_\theta^\alpha(u - \Pi_N^{\sigma,\sigma^*}u)\|_{\omega^{\sigma^*,\sigma}} \le cN^{-m}|\mathcal{L}_\theta^\alpha u|_{B^m_{\omega^{\sigma^*,\sigma}}}. \tag{4.5}$$

Proof. For $\tilde{u} = \omega^{-\sigma,-\sigma^*}u \in B^m_{\omega^{\sigma,\sigma^*}}$, we have the expansion as

$$u = \omega^{\sigma,\sigma^*}\tilde{u} = \omega^{\sigma,\sigma^*}\sum_{n=0}^{\infty} u_n P_n^{\sigma,\sigma^*}. \tag{4.6}$$

By the definition of the operator Π_N^{σ,σ^*}, we obtain

$$u - \Pi_N^{\sigma,\sigma^*}u = \omega^{\sigma,\sigma^*}\sum_{n=N+1}^{\infty} u_n P_n^{\sigma,\sigma^*} = \omega^{\sigma,\sigma^*}(\tilde{u} - \mathcal{P}_N^{\sigma,\sigma^*}\tilde{u}). \tag{4.7}$$

Then by the error estimate for the orthogonal projection $\mathcal{P}_N^{\sigma,\sigma^*}$, e.g. in [26], we obtain (4.4).

By Lemma 2.1, it holds that

$$\mathcal{L}_\theta^\alpha u = \sum_{n=0}^{\infty} \lambda_{\theta,n}^\alpha u_n P_n^{\sigma^*,\sigma}, \quad \mathcal{L}_\theta^\alpha(u - \Pi_N^{\sigma,\sigma^*}u) = \sum_{n=N+1}^{\infty} \lambda_{\theta,n}^\alpha u_n P_n^{\sigma^*,\sigma}.$$

By (A.6), we have

$$D^{(m)}(\mathcal{L}_\theta^\alpha u) = \sum_{n=m}^{\infty} \lambda_{\theta,n}^\alpha d_{n,m}^{\sigma^*,\sigma}\hat{u}_n P_{n-m}^{\sigma^*+m,\sigma+m}.$$

Thus, we have

$$\|\mathcal{L}_\theta^\alpha(u - \Pi_N^{\sigma,\sigma^*}u)\|^2_{\omega^{\sigma^*,\sigma}} = \sum_{n=N+1}^{\infty} (\lambda_{\theta,n}^\alpha)^2|\hat{u}_n|^2 h_n^{\sigma^*,\sigma}, \tag{4.8}$$

$$\|D^{(m)}(\mathcal{L}_\theta^\alpha u)\|^2_{\omega^{\sigma^*+m,\sigma+m}} = \sum_{n=m}^{\infty}(\lambda_{\theta,n}^\alpha)^2(d_{n,m}^{\sigma^*,\sigma})^2|\hat{u}_n|^2 h_{n-m}^{\sigma^*+m,\sigma+m}. \tag{4.9}$$

It follows from the Definition (2.7) that

$$\begin{aligned}
\|\mathcal{L}_\theta^\alpha(u - \Pi_N^{\sigma,\sigma^*}u)\|^2_{\omega^{\sigma^*,\sigma}} &= \sum_{n=N+1}^{\infty} (\lambda_{\theta,n}^\alpha)^2(d_{n,m}^{\sigma^*,\sigma})^2|\hat{u}_n|^2 h_{n-m}^{\sigma^*+m,\sigma+m}\frac{h_n^{\sigma^*,\sigma}}{h_{n-m}^{\sigma^*+m,\sigma+m}}\frac{1}{(d_{n,m}^{\sigma^*,\sigma})^2} \\
&\le \frac{h_{N+1}^{\sigma^*,\sigma}}{h_{N+1-m}^{\sigma^*+m,\sigma+m}}\frac{1}{(d_{N+1,m}^{\sigma^*,\sigma})^2}\sum_{n=N+1}^{\infty}(\lambda_{\theta,n}^\alpha)^2(d_{n,m}^{\sigma^*,\sigma})^2|\hat{u}_n|^2 h_{n-m}^{\sigma^*+m,\sigma+m} \\
&\le \frac{h_{N+1}^{\sigma^*,\sigma}}{h_{N+1-m}^{\sigma^*+m,\sigma+m}}\frac{1}{(d_{N+1,m}^{\sigma^*,\sigma})^2}\|D^{(m)}(\mathcal{L}_\theta^\alpha u)\|^2_{\omega^{\sigma^*+m,\sigma+m}}.
\end{aligned} \tag{4.10}$$

When N is sufficiently large, we have the following asymptotic estimate:

$$\frac{h_{N+1}^{\sigma^*,\sigma}}{h_{N+1-m}^{\sigma^*+m,\sigma+m}}\frac{1}{(d_{N+1,m}^{\sigma^*,\sigma})^2} \le cN^{-2m}. \tag{4.11}$$

Combining (4.8)–(4.11) leads to (4.5). When m is not an integer, we can apply standard space interpolation to obtain the conclusion. □

Lemma 4.2. *Let $\alpha \in (1, 2)$. Suppose u satisfies $\omega^{-\sigma,-\sigma^*}u \in L^2_{\omega^{\sigma,\sigma^*}}$ and $\|\mathcal{L}_\theta^\alpha u\|_{\omega^{\sigma^*,\sigma}} < \infty$. Then we have*

$$\lambda_{\theta,0}^\alpha\|u\|_{\omega^{\sigma^*,\sigma}} \le \lambda_{\theta,0}^\alpha\|u\|_{\omega^{-\sigma,-\sigma^*}} \le \|\mathcal{L}_\theta^\alpha u\|_{\omega^{\sigma^*,\sigma}}. \tag{4.12}$$

Proof. For u satisfying $\omega^{-\sigma,-\sigma^*}u \in L^2_{\omega^{\sigma,\sigma^*}}$, we write $u = \omega^{\sigma,\sigma^*}\sum_{n=0}^{\infty} u_n P_n^{\sigma,\sigma^*}$ and derive from Lemma 2.1 that

$$\|u\|^2_{\omega^{-\sigma,-\sigma^*}} = \sum_{n=0}^{\infty} |u_n|^2 h_n^{\sigma,\sigma^*}, \quad \|\mathcal{L}_\theta^\alpha u\|^2_{\omega^{\sigma^*,\sigma}} = \sum_{n=0}^{\infty} (\lambda_{\theta,n}^\alpha)^2 |u_n|^2 h_n^{\sigma^*,\sigma},$$

where by (2.3), we have $h_n^{\sigma,\sigma^*} = h_n^{\sigma^*,\sigma}$. Noticing the sequence $\{\lambda_{\theta,n}^\alpha\}$ is monotonically increasing, we have

$$\lambda_{\theta,0}^\alpha \|u\|_{\omega^{\sigma^*,\sigma}} \le \lambda_{\theta,0}^\alpha \|u\|_{\omega^{-\sigma,-\sigma^*}} \le \|\mathcal{L}_\theta^\alpha u\|_{\omega^{\sigma^*,\sigma}}.$$

This completes the proof. □

Theorem 4.1 (Stability)**.** *Assume that* $|\mu| \le \lambda_{\theta,0}^\alpha/2$. *The problem (4.1) admits a unique solution* u_N *such that*

$$\|\mathcal{L}_\theta^\alpha u_N\|_{\omega^{\sigma^*,\sigma}} \le C\|f\|_{\omega^{\sigma^*,\sigma}}.$$

Proof. Take $v_N = \omega^{\sigma^*,\sigma}\mathcal{L}_\theta^\alpha u_N$ in (4.1). By Lemma 4.2 and Cauchy–Schwartz inequality, we get

$$\begin{aligned}\|\mathcal{L}_\theta^\alpha u_N\|^2_{\omega^{\sigma^*,\sigma}} &= -\mu(u_N, \omega^{\sigma^*,\sigma}\mathcal{L}_\theta^\alpha u_N) + (f, \omega^{\sigma^*,\sigma}\mathcal{L}_\theta^\alpha u_N)\\ &\le |\mu|\|u_N\|_{\omega^{\sigma^*,\sigma}}\|\mathcal{L}_\theta^\alpha u_N\|_{\omega^{\sigma^*,\sigma}} + \|f\|_{\omega^{\sigma^*,\sigma}}\|\mathcal{L}_\theta^\alpha u_N\|_{\omega^{\sigma^*,\sigma}}\\ &\le |\mu|/\lambda_{\theta,0}^\alpha\|\mathcal{L}_\theta^\alpha u_N\|^2_{\omega^{\sigma^*,\sigma}} + \|f\|_{\omega^{\sigma^*,\sigma}}\|\mathcal{L}_\theta^\alpha u_N\|_{\omega^{\sigma^*,\sigma}}\\ &\le 1/2\|\mathcal{L}_\theta^\alpha u_N\|^2_{\omega^{\sigma^*,\sigma}} + \|f\|_{\omega^{\sigma^*,\sigma}}\|\mathcal{L}_\theta^\alpha u_N\|_{\omega^{\sigma^*,\sigma}},\end{aligned}$$

which leads to the desired result directly. □

Theorem 4.2 (Convergence order)**.** *Suppose that* u *and* u_N *satisfy the problems (3.1) and (4.1), respectively. If* $f \in B^r_{\omega^{\sigma^*-1,\sigma-1}}$ *with* $r \ge 0$ *and* $0 < \mu \le \lambda_{\theta,0}^\alpha/2$, *then we have the following optimal error estimate:*

$$\|u - u_N\|_{\omega^{-\sigma,-\sigma^*}} \le cN^{-\gamma}|\omega^{-\sigma,-\sigma^*}u|_{B^\gamma_{\omega^{\sigma,\sigma^*}}}, \ \gamma = (\alpha+1)\wedge r + \alpha - \epsilon.$$

Proof. Denote $\eta_N = u - \Pi_N^{\sigma,\sigma^*}u$ and $e_N = \Pi_N^{\sigma,\sigma^*}u - u_N$, then $u - u_N = \eta_N + e_N$. Combining (3.1) and (4.1), we can obtain the following error equation:

$$\begin{aligned}(\mathcal{L}_\theta^\alpha e_N, v_N) + \mu(e_N, v_N) &= -(\mathcal{L}_\theta^\alpha \eta_N, v_N) - \mu(\eta_N, v_N),\\ &= -\mu(\eta_N, v_N), \quad \forall v_N \in V_N,\end{aligned}$$

where we have used the orthogonal property

$$(\mathcal{L}_\theta^\alpha(u - \Pi_N^{\sigma,\sigma^*}u), v_N) = \left(\sum_{n=N+1}^{\infty} \lambda_{\theta,n}^\alpha u_n P_n^{\sigma^*,\sigma}, v_N\right) = 0, \quad \forall v_N \in U_N = \omega^{\sigma^*,\sigma}\mathbb{P}_N.$$

Taking $v_N = \omega^{\sigma^*,\sigma}\mathcal{L}_\theta^\alpha e_N$, we get

$$\begin{aligned}\|\mathcal{L}_\theta^\alpha e_N\|^2_{\omega^{\sigma^*,\sigma}} &= -\mu(e_N, \omega^{\sigma^*,\sigma}\mathcal{L}_\theta^\alpha e_N) - \mu(\eta_N, \omega^{\sigma^*,\sigma}\mathcal{L}_\theta^\alpha e_N)\\ &= -\mu(e_N, \omega^{\sigma^*,\sigma}\mathcal{L}_\theta^\alpha e_N) - \mu(\eta_N, \omega^{\sigma^*,\sigma}\mathcal{L}_\theta^\alpha e_N).\end{aligned}$$

Following a similar derivation in the proof for stability Theorem 4.1, we obtain

$$\|\mathcal{L}_\theta^\alpha e_N\|_{\omega^{\sigma^*,\sigma}} \le 2\mu\|\eta_N\|_{\omega^{\sigma^*,\sigma}}.$$

By Lemma 4.2, we have

$$\|e_N\|_{\omega^{-\sigma,-\sigma^*}} \le 1/\lambda_{\theta,0}^\alpha\|\mathcal{L}_\theta^\alpha e_N\|_{\omega^{\sigma^*,\sigma}} \le 2\mu/\lambda_{\theta,0}^\alpha\|\eta_N\|_{\omega^{\sigma^*,\sigma}} \le \|\eta_N\|_{\omega^{-\sigma,-\sigma^*}}.$$

Using the triangle inequality leads to

$$\|u - u_N\|_{\omega^{-\sigma,-\sigma^*}} \le \|e_N\|_{\omega^{-\sigma,-\sigma^*}} + \|\eta_N\|_{\omega^{-\sigma,-\sigma^*}} \le 2\|\eta_N\|_{\omega^{-\sigma,-\sigma^*}}.$$

Since $f \in B^r_{\omega^{\sigma^*-1,\sigma-1}}$ with $r \ge 0$, we can see from Theorem 3.2 that $\omega^{-\sigma,-\sigma^*}u \in B^\gamma_{\omega^{\sigma,\sigma^*}}$. Applying Lemma 4.1 leads to the desired result. □

Remark 4.1. For the spectral Petrov–Galerkin method, we need $0 < \mu \le \lambda_{\theta,0}^\alpha/2$. However, this assumption seems to be relaxed to the case for all $\mu > 0$. The key is to show that $(u_N, \omega^{\sigma^*,\sigma}\mathcal{L}_\theta^\alpha u_N)$ is positive for all $u_N \in U_N$. Unfortunately, this is an open problem and we are not able to prove this for technical reasons. However, when $\theta = 1/2$, the spectral Petrov–Galerkin method coincides with spectral Galerkin method in [46] where we only need $\mu > 0$.

Table A.1
Numerical values for (σ, σ^*) corresponding to different θ and α.

	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
$\theta = 0.7$	(0.8829, 0.3171)	(0.8602, 0.5398)	(0.8900, 0.7100)	(0.9411, 0.8589)
$\theta = 1.0$	(1.0000, 0.2000)	(1.0000, 0.4000)	(1.0000, 0.6000)	(1.0000, 0.8000)

Table A.2
Convergence orders and errors of the spectral Petrov–Galerkin method (4.1) for Example 5.1 with $f = \sin x$.

		$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
	N	$E_1(N)$	rate	$E_1(N)$	rate	$E_1(N)$	rate	$E_1(N)$	rate
$\theta = 0.7$	16	3.84e−05		7.39e−06		1.24e−06		1.75e−07	
	32	3.90e−06	3.30	5.82e−07	3.67	7.68e−08	4.02	8.46e−09	4.37
	64	3.80e−07	3.36	4.40e−08	3.73	4.50e−09	4.09	3.83e−10	4.47
	128	3.60e−08	3.40	3.23e−09	3.77	2.54e−10	4.15	1.66e−11	4.53
Order			3.40		3.80		4.20		4.60
$\theta = 1$	16	3.87e−06		1.99e−06		7.04e−07		1.63e−07	
	32	3.94e−07	3.29	1.56e−07	3.67	4.42e−08	4.00	8.14e−09	4.32
	64	3.80e−08	3.38	1.17e−08	3.74	2.61e−09	4.08	3.75e−10	4.44
	128	3.55e−09	3.42	8.56e−10	3.78	1.49e−10	4.13	1.65e−11	4.51
Order			3.40		3.80		4.20		4.60

5. Numerical results

In this section, we present three examples with different forcing terms f: smooth (Example 1), weakly singular at an interior point (Example 2) and weakly singular at boundary (Example 3).

Since exact solutions are unavailable, we use reference solutions u_{ref}, which are computed with a very fine resolution using the same methods for computing u_N. In the computation, we take $\mu = 1$ and measure the error as follows:

$$E_1(N) = \|u_{\text{ref}} - u_N\|_{\omega^{-2\sigma,-2\sigma^*}}, \quad E_2(N) = \|u_{\text{ref}} - u_N\|_{\omega^{-\sigma,-\sigma^*}} \tag{5.1}$$

where the weight function $\omega^{\gamma,\beta} = (1-x)^\gamma(1+x)^\beta$, $u_N = \sum_{n=0}^N \hat{u}_n \omega^{\sigma,\sigma^*} P_n^{\sigma,\sigma^*}$ and $u_{\text{ref}} =: u_{512}$.

For the first two examples without boundary singularity, we use $E_1(N)$ to measure the error. Note the weighted index is negative, thus $E_2(N) \le E_1(N)$ and the convergence order of $E_1(N)$ is at least the order of $E_2(N)$. For the third example, we use $E_2(N)$ to measure the error.

Here we present numerical results for $\theta \in (0.5, 1]$, in particular, $\theta = 0.7$, and $\theta = 1$. Since σ and σ^* depends on the fractional order α and θ, we find the values of σ, σ^* numerically using Newton's method with a tolerance 10^{-16}. We list in Table A.1 the values of (σ, σ^*) for different θ's and α's. For illustration, we present only four digits in the table while in computation we keep fifteen digits for σ and σ^*.

Example 5.1. Consider $f = \sin x$ for $x \in (-1, 1)$. Here f belongs to $B^{\infty}_{\omega^{\sigma^*-1,\sigma-1}}$. By Theorem 3.2, $\omega^{-\sigma,-\sigma^*}u \in B^{2\alpha+1-\epsilon}_{\omega^{\sigma,\sigma^*}}$.

According to Theorem 4.2, the convergence orders are expected to be $2\alpha + 1 - \epsilon$ for the spectral Petrov–Galerkin method (4.1). In Table A.2, we observe that the convergence orders are $2\alpha + 1$ for the spectral Petrov–Galerkin method when the order $\alpha = 1.2, 1.4, 1.6, 1.8$.

In this example, the spectral Petrov–Galerkin method (4.1) has the convergence order $2\alpha + 1 - \epsilon$, which suggests that the regularity index $2\alpha + 1 - \epsilon$ for the solution and somewhat verifies Theorem 3.2.

Example 5.2. Consider $f = |\sin x|$ for $x \in (-1, 1)$. The function f has a weak singularity at $x = 0$ and $f \in B^{1.5-\epsilon}_{\omega^{\sigma^*-1,\sigma-1}}$ for any $\epsilon > 0$. By Theorem 3.2, $\omega^{-\sigma,-\sigma^*}u \in B^{\alpha+1.5-\epsilon}_{\omega^{\sigma,\sigma^*}}$.

According to Theorem 4.2, the convergence orders are expected to be $\alpha + 1.5 - \epsilon$ for the spectral Petrov–Galerkin method (4.1).

From Table A.3, we can observe that the convergence order is $\alpha + 1.5 - \epsilon$ for the spectral Petrov–Galerkin method (4.1), which is in agreement with the theoretical prediction when the order $\alpha = 1.2, 1.4, 1.6, 1.8$.

In this example, the spectral Petrov–Galerkin method (4.1) has the convergence order $\alpha + 1.5 - \epsilon$, which suggests the regularity index $\alpha + 1.5 - \epsilon$ for the solution and verifies Theorem 3.2.

Table A.3
Convergence orders and errors of the spectral Petrov–Galerkin method (4.1) for Example 5.2 with $f = |\sin x|$.

	N	$\alpha = 1.2$ $E_1(N)$	rate	$\alpha = 1.4$ $E_1(N)$	rate	$\alpha = 1.6$ $E_1(N)$	rate	$\alpha = 1.8$ $E_1(N)$	rate
$\theta = 0.7$	16	1.43e−03		5.05e−04		2.19e−04		1.12e−04	
	32	2.50e−04	2.51	7.73e−05	2.71	2.98e−05	2.87	1.38e−05	3.02
	64	4.00e−05	2.64	1.08e−05	2.84	3.69e−06	3.02	1.52e−06	3.18
	128	6.11e−06	2.71	1.44e−06	2.90	4.33e−07	3.09	1.58e−07	3.26
Order			2.70		2.90		3.10		3.30
$\theta = 1$	16	9.36e−04		4.69e−04		2.32e−04		1.17e−04	
	32	1.69e−04	2.47	7.52e−05	2.64	3.30e−05	2.81	1.47e−05	2.99
	64	2.81e−05	2.59	1.10e−05	2.77	4.25e−06	2.95	1.66e−06	3.15
	128	4.49e−06	2.65	1.54e−06	2.84	5.20e−07	3.03	1.77e−07	3.23
Order			2.70		2.90		3.10		3.30

Table A.4
Convergence orders and errors of the spectral Petrov–Galerkin method (4.1) for Example 5.3 with $f = (1 - x^2)^{-0.4} \sin x$.

	N	$\alpha = 1.2$ $E_2(N)$	rate	$\alpha = 1.4$ $E_2(N)$	rate	$\alpha = 1.6$ $E_2(N)$	rate	$\alpha = 1.8$ $E_2(N)$	rate
$\theta = 0.7$	16	5.75e−03		1.13e−03		3.20e−04		1.12e−04	
	32	1.86e−03	1.63	2.73e−04	2.05	6.05e−05	2.40	1.72e−05	2.71
	64	5.85e−04	1.67	6.37e−05	2.10	1.09e−05	2.47	2.47e−06	2.80
	128	1.80e−04	1.70	1.47e−05	2.12	1.94e−06	2.50	3.45e−07	2.84
Order			1.72		2.14		2.51		2.86
$\theta = 1$	16	4.42e−03		1.16e−03		3.43e−04		1.15e−04	
	32	1.56e−03	1.50	3.15e−04	1.88	7.09e−05	2.27	1.82e−05	2.67
	64	5.33e−04	1.55	8.23e−05	1.94	1.40e−05	2.33	2.71e−06	2.74
	128	1.78e−04	1.58	2.10e−05	1.97	2.72e−06	2.37	3.95e−07	2.78
Order			1.60		2.00		2.40		2.80

Example 5.3 (Boundary singularity for the right hand side f)**.** Consider $f = (1 - x^2)^\beta \sin x$ for $x \in (-1, 1)$. Here $f \in B^{\sigma\wedge\sigma^*+2\beta+1-\epsilon}_{\omega^{\sigma^*,\sigma}}$; see Lemma A.8 in Appendix. From Theorem 3.1, $\omega^{-\sigma,-\sigma^*} u \in B^{\alpha+(\sigma\wedge\sigma^*+2\beta+1)\wedge\alpha-\epsilon}_{\omega^{\sigma,\sigma^*}}$.

We consider the singular forcing term $f = (1 - x^2)^\beta \sin x$ where $\beta = -0.4$. In this case, we apply Theorem 3.1 instead of Theorem 3.2 in order to get higher regularity index. From Theorem 3.1, $\omega^{-\sigma,-\sigma^*} u \in B^{\alpha+(\sigma\wedge\sigma^*+2\beta+1)\wedge\alpha-\epsilon}_{\omega^{\sigma,\sigma^*}}$. For $\beta = -0.4$, $\omega^{-\sigma,-\sigma^*} u \in B^{\alpha+(\sigma\wedge\sigma^*+0.2)\wedge\alpha-\epsilon}_{\omega^{\sigma,\sigma^*}}$. According to Theorem 4.2, the convergence orders for the spectral Petrov–Galerkin method are $\alpha + \sigma \wedge \sigma^* + 0.2 - \epsilon$, which is demonstrated in Table A.4.

6. Summary and discussion

In this paper, we discuss the regularity of the two-sided fractional diffusion equations with Riemann–Liouville operators under the homogeneous Dirichlet boundary conditions. Writing $u = \omega^{\sigma,\sigma^*}\tilde{u}$, we find that the regularity index of $\tilde{u}$ is shown to be $2\alpha + 1 - \epsilon$ with $\epsilon > 0$ arbitrary. We also validate our finding by considering the spectral Petrov–Galerkin method. With the regularity index, we showed the optimal error estimate when the reaction coefficient μ is small.

Our regularity estimate is sharp for $\gamma \le \alpha + 1$ (γ is regularity index of right hand side *f*). One of promising applications of current work is that the regularity estimates can be extended to the low-regularity or non-smooth data *f*, e.g., the fraction differential equations with color or white space-noise.

However, when $\gamma > \alpha + 1$, it is numerically found that the convergence order can reach up to $2\alpha + 1 + \min\{\sigma, \sigma^*\} - \epsilon$, which suggests higher regularity of solution $\tilde{u}$. In the symmetrical case, $\theta = \frac{1}{2}$ and $\sigma = \sigma^* = \frac{\alpha}{2}$, the optimal regularity has been proved in [15]. For the non-symmetrical case, as the equivalence of non-uniformly weighted Sobolev spaces is not established as that in [15], we can not prove the optimal regularity for non-symmetrical case right now. We will address such fundamental issues in our follow-up work.

The estimate for Petrov–Galerkin method requires some additional condition on μ. Further improvement in the error estimates is needed. The analysis in this paper can be extended to FDEs with different low-order terms, such as FDEs with an advection term. We will leave this topic for future research.

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Appendix. Auxiliary lemmas

We denote by $H^s(\Omega)$ and $H_0^s(\Omega)$ the usual Sobolev spaces as in [1] with semi-norm $|\cdot|_{H^s}$

$$|v|_{H^s(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,dxdy\right)^{1/2}.$$

We recall the following results (cf.[8]) which play essential roles in the weak formulation and analysis of fractional differential equations.

Lemma A.1 [8]**.** *For any $v \in H_0^{\alpha/2}$ with $1 < \alpha \le 2$, we have*

$$(\mathcal{L}_\theta^\alpha v, v) = c_1^\alpha |v|^2_{H^{\alpha/2}}, \quad c_1^\alpha = -\cos\left(\frac{\alpha}{2}\pi\right). \tag{A.1}$$

In order to show convergence, we also need Hardy-type inequality below.

Lemma A.2 [25]**.** *Let Λ be a convex set and $1 < \alpha < 2$. For any $v \in C_0^\infty(\Lambda)$, it holds*

$$\int_\Lambda \int_\Lambda \frac{|v(x)-v(y)|^2}{|x-y|^{n+\alpha}}dxdy \ge k_{n,\alpha}\int_\Lambda \frac{|v(x)|^2}{d_\Lambda(x)^\alpha}dx, \tag{A.2}$$

where C and $k_{n,\alpha}$ are positive constants which only depend on dimension n, α, and $d_\Lambda(x)$ denotes the distance from point $x \in \Lambda$ to the boundary of Λ.

By Lemma A.2, it can be readily shown that the inequality (A.2) still holds for $v \in H_0^{\alpha/2}$ with $\alpha \in (1, 2)$, i.e.,

$$\|v\|_{\omega^{-\alpha,-\alpha}} \le C|v|_{H^{\alpha/2}}, \quad \forall v \in H_0^{\alpha/2}. \tag{A.3}$$

The inequality (A.3) immediately leads to the following conclusion:

Lemma A.3. *For any $v \in H_0^{\alpha/2}$ with $1 < \alpha \le 2$, we have*

$$\|v\|_{\omega^{-\sigma,-\sigma^*}} \le C|v|_{H^{\alpha/2}}, \quad \|v\|_{\omega^{-\sigma^*,-\sigma}} \le C|v|_{H^{\alpha/2}}, \tag{A.4}$$

where σ and σ^* are defined in Lemma 2.1.

The following relations hold for Jacobi polynomials $P_n^{\alpha,\beta}(x)$, see e.g. [2],

$$\partial_x P_n^{\alpha,\beta}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{\alpha+1,\beta+1}(x), \quad \alpha,\beta > -1. \tag{A.5}$$

By (A.5), we have

$$\partial_x^l P_n^{\alpha,\beta}(x) = d_{n,l}^{\alpha,\beta}P_{n-l}^{\alpha+l,\beta+l}(x), \quad \alpha,\beta > -1, n \ge l, \quad d_{n,l}^{\alpha,\beta} = \frac{\Gamma(n+\alpha+\beta+l+1)}{2^l\Gamma(n+\alpha+\beta+1)}. \tag{A.6}$$

Theorem A.1. *The Jacobi polynomials $P_n^{\alpha,\beta}(x)$ satisfy*

$$P_n^{\alpha,\beta} = \widehat{A}_n^{\alpha,\beta}\partial_x P_{n-1}^{\alpha,\beta} + \widehat{B}_n^{\alpha,\beta}\partial_x P_n^{\alpha,\beta} + \widehat{C}_n^{\alpha,\beta}\partial_x P_{n+1}^{\alpha,\beta}, \; n \ge 0, \tag{A.7}$$

where $P_{-1}^{\alpha,\beta} \equiv 0$ and

$$\widehat{A}_n^{\alpha,\beta} = \frac{-2(n+\alpha)(n+\beta)}{(n+\alpha+\beta)(2n+\alpha+\beta)(2n+\alpha+\beta+1)},$$

$$\widehat{B}_n^{\alpha,\beta} = \frac{2(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \qquad \widehat{C}_n^{\alpha,\beta} = \frac{2(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}.$$

The relation (A.5) and Theorem A.1 lead to the following result:

Corollary A.1. *The Jacobi polynomials $P_n^{\alpha,\beta}(x)$ satisfy*

$$P_n^{\alpha,\beta} = A_n^{\alpha,\beta}P_{n-2}^{\alpha+1,\beta+1} + B_n^{\alpha,\beta}P_{n-1}^{\alpha+1,\beta+1} + C_n^{\alpha,\beta}P_n^{\alpha+1,\beta+1}, \; n \ge 0, \tag{A.8}$$

where we let $A_0^{\alpha,\beta}=A_1^{\alpha,\beta}=B_0^{\alpha,\beta}=0$ *and* $P_{-2}^{\alpha+1,\beta+1}=P_{-1}^{\alpha+1,\beta+1}=0$ *and*

$$A_n^{\alpha,\beta}=-\frac{(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)},\quad B_n^{\alpha,\beta}=\frac{(\alpha-\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$C_n^{\alpha,\beta}=\frac{(n+\alpha+\beta+1)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}.$$

Lemma A.4. *For any* $k\ge n\ge 0$*, it holds that* $\left|X_k^n\right|\le C$ *where*

$$X_k^n:=\frac{(P_k^{\alpha+1,\beta+1},P_n^{\alpha,\beta})_{\omega^{\alpha,\beta}}}{h_n^{\alpha,\beta}},\quad \alpha>-1,\quad \beta>-1. \tag{A.9}$$

and $h_n^{\alpha,\beta}$ *is defined in (2.3).*

Proof. By (3.2), we get

$$\delta_{nk}=A_k^{\alpha,\beta}X_{k-2}^n+B_k^{\alpha,\beta}X_{k-1}^n+C_k^{\alpha,\beta}X_k^n. \tag{A.10}$$

Thus we have

$$X_n^n=\frac{1}{C_n^{\alpha,\beta}},\quad X_{n+1}^n=-\frac{B_{n+1}^{\alpha,\beta}}{C_{n+1}^{\alpha,\beta}}X_n^n,\quad X_{k+2}^n=p_kX_{k+1}^n+q_kX_k^n,\quad k\ge n.$$

where $p_k=-\frac{B_{k+2}^{\alpha,\beta}}{C_{k+2}^{\alpha,\beta}}$ and $q_k=-\frac{A_{k+2}^{\alpha,\beta}}{C_{k+2}^{\alpha,\beta}}$. Denote $Y_k^n=(X_{k+1}^n,X_k^n)^\top$ and $A_k=(p_k,q_k;1,0)$. Then we have $Y_{k+1}^n=A_kY_k^n$. It follows that

$$\|Y_{k+1}^n\|_\infty=\|A_kY_k^n\|_\infty\le\|A_k\|_\infty\|Y_k^n\|_\infty=\max\{|1,|p_k|+q_k\}\|Y_k^n\|_\infty,$$

where $\|Y_k^n\|_\infty=\max\{|X_k^n|,|X_{k+1}^n|\}$. Recalling $A_k^{\alpha,\beta}$, $B_k^{\alpha,\beta}$ and $C_k^{\alpha,\beta}$ in Corollary A.1, we have for $k\ge 2$

$$|p_{k-2}|=\frac{|B_k^{\alpha,\beta}|}{C_k^{\alpha,\beta}}=\frac{|\alpha-\beta|(2k+\alpha+\beta+1)}{(2k+\alpha+\beta)(k+\alpha+\beta+2)}=\frac{|\alpha-\beta|}{k}+\mathcal{O}\left(\frac{1}{k^2}\right),\quad \text{and}$$
$$q_{k-2}=-\frac{A_k^{\alpha,\beta}}{C_k^{\alpha,\beta}}=\frac{(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)}{(k+\alpha+\beta+1)(k+\alpha+\beta+2)(2k+\alpha+\beta)}$$
$$=1-\frac{\alpha+\beta+2}{k}+\mathcal{O}\left(\frac{1}{k^2}\right).$$

Thus we arrive at

$$|p_k|+q_k=1-\frac{\alpha+\beta+2-|\alpha-\beta|}{k}+\mathcal{O}\left(\frac{1}{k^2}\right)=1-\frac{2\min(\alpha,\beta)+2}{k}+\mathcal{O}\left(\frac{1}{k^2}\right).$$

Since $\alpha>-1$ and $\beta>-1$, $|p_k|+q_k\le 1$. Thus for any $k\ge n\ge 0$, we have $\|Y_{k+1}\|_\infty\le\|Y_k\|_\infty$, which leads to the desired result. □

In the proof of the regularity of the problem (1.1) and (1.2), we have used the following lemmas. We refer the interested readers to [14] for proofs of lemmas.

Lemma A.5. *Let* $u=v\omega^{\sigma,\sigma^*}$*. If* $v\in B^1_{\omega^{\sigma,\sigma^*}}$*. then* $u\in B^1_{\omega^{\sigma^*,\sigma}}$*. If* $v\in B^k_{\omega^{\sigma-1,\sigma^*-1}}$ *for* $k=1,2$*, then* $u\in B^k_{\omega^{\sigma^*-1,\sigma-1}}$*.*

Lemma A.6. *Let* $u=v\omega^{\sigma,\sigma^*}$ *with* $v\in B^2_{\omega^{\sigma,\sigma^*}}$*. Then* $u\in B^{\alpha-\epsilon}_{\omega^{\sigma^*,\sigma}}$ *with* $\epsilon>0$ *arbitrary and* $1<\alpha<2$*.*

Lemma A.7. *Let* $u=v\omega^{\sigma,\sigma^*}$ *with* $v\in B^3_{\omega^{\sigma-1,\sigma^*-1}}$*. Then* $u\in B^{\alpha+1-\epsilon}_{\omega^{\sigma^*,\sigma}}$ *with* $\epsilon>0$ *arbitrary and* $1<\alpha<2$*.*

Lemma A.8 (c.f. Example 5.3 in Section 5)**.** *Let* $u=(1-x^2)^\beta\phi(x)$ *with* $\beta>-1/2$*. Here* $\phi(x)$ *is* C^∞ *function. Then* $u\in B^{2\beta+\sigma\wedge\sigma^*+1-\epsilon}_{\omega^{\sigma^*,\sigma}}$ *with* $\epsilon>0$*.*

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