LECTURE 9 STOCHASTIC PARABOLIC EQUATIONS AND THEIR NUMERICAL METHODS: II. INFINITE DIMENSIONAL NOISES, NONLINEAR PROBLEMS

1. Stochastic parabolic equations with additive noise

Let us discuss the regularity of a simple SPDE – one-dimensional heat equation with additive space-time noise. As will be shown, the regularity is low and depends on the smoothness of the driving space-time noise.

Define the following cylindrical process on a bounded domain $\mathcal{D}$.

$$\dot{W}^Q(t,x) = \sum_{k=1}^{\infty} \sqrt{q_k} m_k(x) \dot{W}_k(t), \quad x \in \mathcal{D} \subsetneq \mathbb{R}^d. \tag{1.1}$$

Here $\{m_k(x)\}_k$ is a complete orthonormal basis in $L^2([0,l])$ and W_k's are i.i.d. standard Brownian motion. Here q_k's are the eigenvalues of the self-adjoint compact operator Q and $m_k(x)$ are the corresponding eigenfunctions (thus $\{m_k(x)\}_{k\geq 1}$ forms a complete orthonormal basis in $L^2(\mathcal{D})$). When $Q = I$, the process is called the space-time white noise.

Remark 1.1. *We can also think $Q(x,y) = \sum_{k=1}^{\infty} q_k m_k(x) m_k(y)$ when $Q(x,y)$ is a positive-definite covariance function on $\mathcal{D} \times \mathcal{D}$. In this case, $\mathbb{E}[W^Q(t,x)W^Q(s,y)] = Q(x,y) t \wedge s$ and $\mathbb{E}[\dot{W}^Q(t,x)\dot{W}^Q(t,x)] = Q(x,y)\delta(t-s)$.*

Example 1.2 (Heat equation with random forcing)**.** *Consider the following one dimensional heat equation driven by some space-time forcing:*

$$\partial_t u(t,x) = \nu \partial_x^2 u(t,x) + \dot{W}^Q(t,x), \quad (t,x) \in (0,\infty) \times \mathcal{D},\ \mathcal{D} = (0,l), \tag{1.2}$$

with vanishing Dirichlet boundary conditions and deterministic initial condition $u_0(x)$. Here $\nu > 0$ is a physical constant depending on the conductivity of the thin wire and $\dot{W}^Q(t,x)$ is defined in (1.1) *with $\sum_{k=1}^{\infty} q_k < \infty$. Also, we assume that Q has the same eigenfunctions as $-\partial_x^2$.*

To find a solution, we apply the *method of eigenfunction expansion.* Let $\{e_k(x)\}_k$ be eigenfunctions of the operator ∂_x^2 with vanishing Dirichlet boundary conditions) and the corresponding eigenvalues are λ_k:

$$-\partial_x^2 e_k = \lambda_k e_k, \quad e_k|_{x=0,l} = 0, \quad k = 1, 2, \ldots. \tag{1.3}$$

with $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ and $\lim_{k\to\infty} \lambda_k = +\infty$. Actually, they can be computed explicitly

$$\lambda_k = k^2 (\frac{\pi}{l})^2, \quad e_k(x) = \sqrt{\frac{2}{l}} \sin(k\frac{\pi}{l}x). \tag{1.4}$$

For simplicity, we let $\nu = 1$ for simplicity. and $m_k = e_k$ for any k. We look for a formal solution of the following form

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) e_k(x).$$

Date: November 12, 2019.

Plugging this formula into (1.2) and multiplying by e_i before integrating over both sides of the equation, we have

$$du_i(t) = -\lambda_i u_i(t)\,dt + \sqrt{q_i}\,dW_i(t), \quad u_i(0) = \int_0^l u_0(x)e_i(x)\,dx.$$

This is the Ornstein-Uhlenbeck process and the solution is

$$u_i(t) = u_{0,i}e^{-\lambda_i t} + \sqrt{q_i}\int_0^t e^{-\lambda_i(t-s)}\,dW_i(s), \quad u_{0,i} = \int_0^l u_0(x)e_i(x)\,dx.$$

Thus the solution is

$$u(t,x) = \sum_{k=1}^{\infty}[\int_0^l u_0(x)e_k(x)\,dx e^{-\lambda_k t} + \sqrt{q_k}\int_0^t e^{-\lambda_k(t-s)}\,dW_k(s)]e_k(x). \tag{1.5}$$

1.1. **Regularity of the solution.** When $\sum_{k=1}^{\infty} q_k < \infty$, we have $\mathbb{E}[\|u(t,x)-u(s,x)\|^2] \le C(t-s)$ ($t-s$ small) and then the solution is Hölder continuous in time, by Kolmogorov's continuity theorem. In fact,

$$\begin{aligned}
\mathbb{E}[\|u(t,x)-u(s,x)\|^2] &= \sum_{k=1}^{\infty}\mathbb{E}[|u_k(t)-u_k(s)|^2] \\
&= \sum_{k=1}^{\infty}\left|u_{0,k}\big(e^{-\lambda_k t}-e^{\lambda_k s}\big)\right|^2 + q_k\mathbb{E}[\left|\int_0^t e^{-\lambda_k(t-\theta)}\,dW_k(\theta) - \int_0^s e^{-\lambda_k(s-\theta)}\,dW_k(\theta)\right|^2] \\
&\le \sum_{k=1}^{\infty}[\lambda_k\left|u_{0,k}\right|^2(t-s)^2 + q_k\frac{1-e^{-2\lambda_k(t-s)}}{\lambda_k}(t-s)] \\
&\le C(t-s)^2\sum_{k=1}^{\infty}\left|u_{0,k}\right|^2\lambda_k + (t-s)\sum_{k=1}^{\infty}q_k\frac{1-e^{-2\lambda_k(t-s)}}{\lambda_k} \le C(t-s),
\end{aligned}$$

where we require that $u_0 \in H^1([0,l])$, i.e., $\sum_{k=1}^{\infty} u_{0,k}^2 k^2 < \infty$. In the second last line, we also have used the following conclusion (left as an exercise)

$$\mathbb{E}[\left|\int_0^t e^{-\lambda(t-\theta)}\,dW(\theta) - \int_0^s e^{-\lambda(s-\theta)}\,dW(\theta)\right|^2] \le \frac{1-e^{-2\lambda(t-s)}}{\lambda}, \text{ for any } t,\ \lambda > 0. \tag{1.6}$$

We can show that the solution is Hölder continuous with exponent less than 1. By the fact $|e_k(x)-e_k(y)| \le \sqrt{\frac{2}{l}}k\frac{\pi}{l}\,|x-y|$, it can be readily checked that

$$\begin{aligned}
\mathbb{E}[|u(t,x)-u(t,y)|^2] &= \sum_{k=1}^{\infty}\mathbb{E}[|u_k(t)|^2]\,|e_k(x)-e_k(y)|^2 \\
&\le \frac{2\pi^2}{l^3}\,|x-y|^2\sum_{k=1}^{\infty}\mathbb{E}[|u_k(t)|^2]k^2 \\
&= \frac{2\pi^2}{l^3}\,|x-y|^2\sum_{k=1}^{\infty}\big(\left|u_{0,k}^2\right|e^{-2\lambda_k t} + q_k\frac{1-e^{-2\lambda_k t}}{2\lambda_k}\big)k^2.
\end{aligned}$$

Recalling (1.4) and $u_0 \in H^1([0,l])$, we have

$$\mathbb{E}[\|u(t,x)-u(t,y)\|^2] \le \big(C + \sum_{k=1}^{\infty} q_k\big)\,|x-y|^2. \tag{1.7}$$

The regularity in x follows from Kolmogorov's continuity theorem.

By the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}[\sup_{0\le t\le T}|u^p(t,x)|] \le C_p \left|\sum_{k=1}^{\infty} q_k e_k(x)\frac{1-e^{-2\lambda_k t}}{2\lambda_k}\right|^{p/2}, \quad p\ge 1. \tag{1.8}$$

As long as $\sum_{k=1}^{\infty}\frac{q_k}{\lambda_k}$ converges, $\mathbb{E}[\sup_{0\le t\le T}|u^p(t,x)|]<\infty$. However, the second-order derivative of the solution in x should be understood as a distribution instead of a function. For simplicity, let's suppose that $u_0(x)=0$. The solution becomes

$$u(t,x)=\sum_{k=1}^{\infty}\sqrt{q_k}\int_0^t e^{-\lambda_k(t-s)}\,dW_k(s)e_k(x). \tag{1.9}$$

The second derivative of $u(t,x)$ in x is

$$\partial_x^2 u(t,x)=\sum_{k=1}^{\infty}\lambda_k\sqrt{q_k}\int_0^t e^{-\lambda_k(t-s)}\,dW_k(s)e_k(x). \tag{1.10}$$

As a Gaussian process, this process exists a.s. and requires a bounded second-order moment, i.e.,

$$\mathbb{E}[\big(\partial_x^2 u(t,x)\big)^2]=\sum_{k=1}^{\infty}q_k\lambda_k\frac{1-e^{-2\lambda_k t}}{2}e_k(x)\ge\frac{1-e^{-2\lambda_1 t}}{2}\sum_{k=1}^{\infty}q_k\lambda_k e_k(x).$$

Thus if q_k is proportional to $1/k^p$, $p<3$, $\sum_{k=1}^{\infty}q_k\lambda_k$ diverges. The condition on $\sum_{k=1}^{\infty}q_k<\infty$ will not give us second-order derivatives in a classical sense.

In conclusion, the solution to (1.2) is not smooth and in general, does not have second-order derivatives unless the space-time noise is very smooth in space. For example, when $q_k=0$ for $k\ge N>1$, we have a finite-dimensional noise, we can expect second-order derivatives in space.

Exercise 1.3. *Show that when $Q=I$, the solution is Hölder continuous, in time with exponent $1/4-\epsilon$ and in physical space with exponent $1/2-\epsilon$.*

1.2. **Numerical methods.** Let's first consider a semi-discretization of (1.2).

$$d\tilde{u}=\nu\partial_x^2\tilde{u}\,dt+\,dW^{Q,N}(t,x), \tag{1.11}$$

where

$$\dot{W}^{Q,N}(t,x)=\sum_{k=1}^{N}\sqrt{q_k}m_k(x)\dot{W}_k(t). \tag{1.12}$$

Recall that

$$u(t)=\int_{\mathcal{D}}G_t(x,y)u_0(y)\,dy+\int_0^t\int_{\mathcal{D}}G_{t-s}(x,y)\,dW^Q(s,y)$$

where $G_t(x,y)$ is the Green function. Then

$$\begin{aligned}
\mathbb{E}[\|u-\tilde{u}\|^2] &= \mathbb{E}[\left\|\int_0^t\int_{\mathcal{D}}G_{t-s}(x,y)\,d(W^Q(s,y)-W^{Q,N}(s,y))\right\|^2] \\
&= \mathbb{E}[\left\|\sum_{k=N+1}^{\infty}\sqrt{q_k}\int_0^t\int_{\mathcal{D}}G_{t-s}(x,y)e_k(y)\,dy\,dW_k(s)\right\|^2] \\
&= \sum_{k=N+1}^{\infty}q_k\,\|e_k(y)\|^2\int_0^t e^{-2\lambda_k(t-s)}\,ds=\sum_{k=N+1}^{\infty}q_k\frac{1-e^{-2\lambda_k t}}{2\lambda_k}\le\sum_{k=N+1}^{\infty}\frac{q_k}{2\lambda_k}.
\end{aligned}$$

When $q_k \equiv 1$, then $\sum_{k=N+1}^{\infty} \frac{q_k}{2\lambda_k} = \sum_{k=N+1}^{\infty} \frac{1}{2(\delta t)^2\pi^2} \leq \frac{1}{2\pi^2} N^{-1}$. Thus, we obtain a half-order convergence in physical space.

Fully discretization. Suppose that $u_0 \in H^1([0,l])$. Let's consider the θ-scheme

$$u_\theta^{n+1} = u_\theta^n + \delta t \nu D_2(\theta u_\theta^{n+1} + (1-\theta)u_\theta^n) + \sigma P_h \Delta W_n^{Q,N}, \quad n = 0, 1, 2, \dots \tag{1.13}$$

Here $(P_h \Delta W_n^Q)_j = \sum_{k=1}^N \sqrt{q_k} e_k(x_j)(W_k(t_{n+1}) - W_k(t_n))$ and N is proportional to $1/h$. Then

$$u_\theta^{n+1} = (I - \nu\delta t\theta D_2)^{-1}(I + \nu\delta t(1-\theta)D_2)u_\theta^n + (I - \nu\delta t\theta D_2)^{-1}\sigma P_h \Delta W_n^{Q,N},$$

and we further have

$$u_\theta^{n+1} = \mathfrak{A} u_\theta^n + \sigma \mathfrak{B} P_h \Delta W_n^Q = \mathfrak{A}^{n+1} u^0 + \sum_{l=0}^n \mathfrak{A}^{n-l} \mathfrak{B} P_h \Delta W_l^Q.$$

where

$$\mathfrak{A} \;=\; (I - \nu\delta t\theta D_2)^{-1}(I + \nu\delta t(1-\theta)D_2), \quad \mathfrak{B} = \sigma(I - \nu\delta t\theta D_2)^{-1}.$$

Recall that

$$u(t) = \int_{\mathcal{D}} G_t(x,y)u_0(y)\,dy + \int_0^t \int_{\mathcal{D}} G_{t-s}(x,y)\,dW^Q(s,y)$$

Then we can write

$$u(t_{n+1}) = G_{t_{n+1}} u_0 + \sum_{l=0}^n \int_{\mathcal{D}} G_{t_{n+1}-t_{l+1}}(x,y) \int_{t_l}^{t_{l+1}} G_{t_{l+1}-s}(x,y)\,dW^Q(s,y).$$

Then the error $e^{n+1}(x_j) = u_{\theta,j}^{n+1} - u(t_{n+1}, x_j)$ may be represented as

$$\begin{aligned} e^{n+1} \;=\; & (\mathfrak{A}^{n+1}u^0 - P_h \int_{\mathcal{D}} G_{t_{n+1}}(x,y)u_0(y)\,dy) \\ & + \sum_{l=0}^n (\mathfrak{A}^{n-l} - P_h G^h_{t_{n+1}-t_{l+1}})(\int_{t_l}^{t_{l+1}} G_{t_{l+1}-s}\,dW^Q(s) \\ & + \sum_{l=0}^n \mathfrak{A}^{n-l}(\mathfrak{B} P_h \Delta W_l^{Q,N} - P_h \int_{t_l}^{t_{l+1}} G_{t_{l+1}-s}\,dW^Q(s). \end{aligned}$$

Here $P_h\cdot$ means the projection of "$\cdot$" at the grid points.

Exercise 1.4. *Apply the facts in Appendix to work out the error estimate when $\theta \in [0,1]$. When $\theta = 1/2$, the convergence order is expected to be $(\delta t)^{1/4} + (\frac{(\delta t)^2}{(\delta x)^3})^{1/2}$.*

The conclusion suggests that when δt is larger than $(\delta x)^{3/2}$, the Crank-Nicolson scheme is not convergent! Observe that this extra condition $\delta t < (\delta x)^{3/2}$ is not from stability requirement but for convergence. The reason for this constraint is that the Crank-Nicolson scheme has difficulty working with high-frequency modes as

$$\frac{1 - \lambda\delta t/2}{1 + \lambda\delta t/2}, \tag{1.14}$$

is around -1, when $\lambda\delta t_0 \gg 1$ and δt_0 is some step size that we can deploy on a computer (with an affordable budget). For example, we can consider

$$y' = -10^{11}(y - \sin(t)) + \cos(t).$$

With time step sizes 10^{-3}, 10^{-4}, 10^{-5}, we can not reach accuracy of second-order as the Crank-Nicolson scheme ($\theta = 1/2$) cannot damp the errors for high frequency modes. In contrast, the backward Euler scheme can damp the errors for high frequency modes quickly as $(1+\delta t\lambda)^{-1}$ is close to 0 when $\lambda\delta t_0 \gg 1$.

This issue of the Crank-Nicolson scheme is well known in numerical methods for parabolic equations with a rough initial condition; see [Thomée, 2006]. To remove the extra constraint on the ratio of step sizes of the Crank-Nicolson scheme, we may use the following modified theta-scheme

$$u_\theta^1 = u_\theta^n + \delta t\nu D_2 u_\theta^{n+1} + \sigma_1\sigma P_h \Delta W_0^{Q,N}, \tag{1.15}$$

$$u_\theta^{n+1} = u_\theta^n + \delta t\nu D_2(\theta u_\theta^{n+1} + (1-\theta)u_\theta^n) + \sigma P_h\Delta W_n^{Q,N}, \quad n = 1,2,\dots \tag{1.16}$$

Here σ_1 can be 0 or 1.

Another solution is to take $\theta = 1/2 + \delta t$, but this will impose some mild constraint on the ratio of step sizes in t and x. (See the next section for error estimates for parabolic SPDEs with multiplicative noise)

Remark 1.5. *Another approach of discretizing W^I is to use a piecewise constant process. We can also approximate the space-time white noise by using piecewise constant functions on a partition of $[0,T]\times[0,1]$: say $[t_{n-1},t_n]\times[x_{j-1},x_j]$, $1\le n\le N, 1\le j\le J$ with $\delta t = t_n - t_{n-1}$ and $\delta x = x_j - x_{j-1}$. More precisely,*

$$\frac{dW(t,x)}{dtdx} \approx \frac{d\hat{W}(t,x)}{dtdx} = \frac{\partial^2\hat{W}(t,x)}{\partial t\partial x} = \frac{1}{\delta t\delta x}\sum_{n=1}^{N}\sum_{j=1}^{J}\xi_{n,j}\sqrt{\delta t\delta x}\,\chi_{[t_{n-1},t_n]}(t)\chi_{[t_{j-1},t_j]}(x), \tag{1.17}$$

where

$$\eta_{nj} = \frac{1}{\delta t\delta x}\int_{t_{n-1}}^{t_n}\int_{x_{j-1}}^{x_j} dW(t,x) = \mathcal{N}(0,1). \tag{1.18}$$

2. Multiplicative noise

Example 2.1 (Multiplicative noise)**.**

$$du = \nu u_{xx}dt + \sigma u\, dW^Q(t,x), \quad x\in(0,1)$$

with a certain boundary condition. When $Q = I$, then the solution is Hölder continuous in time with exponent $1/4-\epsilon$ and in physical space with exponent $1/2-\epsilon$.

We can work with the mild solution. Suppose that $G_t(x-y)$ is the fundamental solution of the problem $\partial_t v = a\partial_x^2 v$ with a certain boundary condition. The solution can be written as

$$u = \int_0^{2\pi} G_t(x,y)u_0(y)\,dy + \int_0^t\int_0^{2\pi} G_{t-s}(x,y)u(y)\,dW^Q(s,y) \tag{2.1}$$

Here

$$G_t(x,y) = \sum_{k=1}^{\infty} e^{-\nu\lambda_k t}e_k(x)e_k(y), \quad (t,x)\in(0,T]\times\bar{\mathcal{D}}. \tag{2.2}$$

and $\partial_x^2 e_k(x) = -\lambda_k e_k(x)$ with the given boundary condition. When a vanishing boundary condition is given, $e_k(x) = \sqrt{2}\sin(k\pi x)$, $\lambda_k = k^2\pi^2$; when the vanish Neumann boundary condition (or a periodic boundary condition) is given, $e_k(x) = \sqrt{2}\cos((k-1)\pi x)$ ($k\ge 2$; $e_1(x) = 1$), $\lambda_k = (k-1)^2\pi^2$ (the solution is unique up to a constant).

Example 2.2 (Multiplicative noise)**.**

$$du = au_{xx}dt + \sigma u_x\, dW^Q(t,x), \quad x \in (0, 2\pi)$$

with a certain boundary condition. What is the condition for the solution to be square-integrable in both random and physical space?

The mild solution can be written as

$$u = \int_0^{2\pi} G_t(x,y)u_0(y)\,dy + \int_0^t \int_0^{2\pi} G_{t-s}(x,y)\partial_y u(y)\,dW^Q(s,y) \tag{2.3}$$

Here $G_t(x,y)$ is given in (2.2).

Example 2.3. *Consider the following problem with space-time white noise.*

$$\begin{cases} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + g(U)\dot{W}^Q + f(U), & (x,t) \in (0,L) \times (0,\infty) \\ U(x,0) = u_0(x), & x \in (0,L) \\ U(0,t) = U(L,t) = 0, & t > 0 \end{cases} \tag{2.4}$$

The mild solution of this problem can be written as

$$\begin{aligned} U(x,t) = &\int_{L_0} G_t(x,y)u_0(y)\mathrm{d}y + \int_0^t \int_0^L G_{t-s}(x,y)g(U(y,s))dW^Q(s,y)+ \\ &+ \int_0^t \int_0^L G_{t-s}(x,y)f(U(y,s))\,dy\mathrm{d}s \end{aligned} \tag{2.5}$$

Here $G_t(x,y)$ is the Green's function or fundamental solution for the homogeneous equation (i.e., the solution when $f = 0 = g$ and $u_0(x) = \delta(x-y)$).

2.1. **Numerical methods: the θ-scheme.** Let's consider a family of semi-implicit methods, called one-step theta schemes for (2.4). The scheme reads [Walsh, 2005]

$$\tilde{U}_\theta^{n+1} = \tilde{U}_\theta^n + \delta t A(\theta \tilde{U}_\theta^{n+1} + (1-\theta)\tilde{U}_\theta^n) + \bar{f}(\tilde{U}_\theta^n)\Delta W_n^I + \delta t \bar{g}(\tilde{U}_\theta^{n+1}). \tag{2.6}$$

Here $(\bar{f}(v))_j = \frac{F(v^{j+1}) - F(v^j)}{v^{j+1} - v^j}$ and $F(\cdot)$ is the anti-derivative of $f(\cdot)$. The best convergence rate of this scheme is $O((\delta x))^{1/2} + (\delta t)^{1/4})$.

Theorem 2.4 ([Walsh, 2005])**.** *Let f and g be Lipschitz continuous functions on the line and let U be the solution of (2.4). Let $\tilde{U}_\theta$ be the numerical approximation given by the one-step theta scheme with space step size δx and time-step size δt. Suppose $u_0 \in H^1([0,L])$.*

$$\sup_{x\in[0,L], t\leq T} \left(\mathbb{E}[\left(U_t(x) - \tilde{U}_t^\theta(x)\right)^2]\right)^{1/2} \leq \begin{cases} C_T(\delta x)^{1/2} & \text{if } \theta = 0 \\ C_T\left((\delta x)^{1/2} + \left(\frac{(\delta t)^2}{(\delta t)^3}\right)^{1/2}\right) & \text{if } \theta = 1/2 \\ C_T\left((\delta x)^{1/2} + \left(\frac{\delta t}{\delta x}\right)^{1/2}\right) & \text{if } 1/2 < \theta < 1 \\ C_T\left((\delta x)^{1/2} + (\delta t)^{1/4}\right) & \text{if } \theta = 1 \end{cases} \tag{2.7}$$

Remark 2.5 (Almost sure convergence)**.** *Suppose that $0 \leq \theta \leq 1$. Taking $(\delta x) \sim 2^{-m}L$ and that $\delta t/(\delta x)^2$ is bounded. Then for any $x \in (0,L)$, $t \leq T$, and $\varepsilon > 0$, it follows from Borel-Cantelli Lemma that*

$$|U(x,t) - \tilde{U}(x,t)|(\delta x)^{\varepsilon - 1/2} \longrightarrow 0, \text{ a.s. and a.e..} \tag{2.8}$$

We may observe the following, which are significantly different from numerical methods for PDEs.

- The θ-schemes behave differently if δt is large compared to $(\delta x)^2$.

- Though the schemes with $\theta \geq 1/2$ are stable for any value of δt, this is not sufficient to guarantee convergence. Indeed, it can be shown that
  - if $\delta t \geq \delta x$, the schemes need not converge when $1/2 < \theta < 1$;
  - when $\theta = 1/2$ (the Crank-Nicholson scheme), its error is $O((\delta x)^{1/2} + \delta t/(\delta x)^{3/2})$, and one needs $\delta t = o((\delta x)^{3/2})$ to guarantee the convergence.
- (Rule of thumb) The step sizes $(\delta x)^2$ and δt should be roughly of the same magnitude for stability *and accuracy.*

As for additive noise, remedy for Crank-Nicholson scheme is simple: compute the numerical solution in the first few steps using the backward Euler scheme and then continue with the Crank-Nicholson scheme. The intuition is that high-frequency modes do not decay quickly in the Crank-Nicholson scheme while these modes are heavily-weighted in the stochastic solution while the backward Euler damps out high-frequency modes very quickly and gives the optimum rate of convergence for any δt and δx.

3. More numerical methods

Let's consider the following SPDE over the physical domain $\mathcal{D} \subseteq \mathbb{R}$,

$$dX = [\mathcal{A}X + f(X)]\, dt + g(X)\, dW^Q, \tag{3.1}$$

where the Q-Wiener process W^Q is defined in (1.1). The leading operator, $\mathcal{A}$, can be a second-order or fourth-order differential operator, which is positive definite. The nonlinear functions f, g are Lipschitz continuous. The problem (3.1) is endowed either with only initial conditions in the whole space ($\mathcal{D} = \mathbb{R}$) or with initial and boundary conditions in a bounded domain ($\mathcal{D} \subsetneq \mathbb{R}$).

3.1. **Direct semi-discretization methods for parabolic SPDEs: discretization in space first.** The *time-discretization* methods for (3.1) can be seen as a straightforward application of numerical methods for SODEs, where increments of Brownian motions are used. After performing a truncation in physical space, we will obtain a system of finite-dimensional SODEs, and subsequently, we can apply standard numerical methods for SODEs, e.g., those from [Kloeden and Platen, 1992, Milstein, 1995, Milstein and Tretyakov, 2004]. It is very convenient to extend the methods of numerical PDEs and SODEs to solve SPDEs. One can select the optimal numerical methods for the underlying SPDEs with carefully analyzing the characteristics of related PDEs and SODEs.

However, it is not possible to derive high-order schemes with direct time discretization methods as the solutions to SPDEs have very low regularity (when the noise is approximated in a naive way). For example, for the heat equation with additive space-time white noise in one dimension (1.2), the sample paths of the solution is Hölder continuous with exponent $1/4 - \epsilon$ ($\epsilon > 0$ is arbitrarily small) in time and is Hölder continuous with exponent $1/2 - \epsilon$ in space.

If the infinite-dimensional noise (1.1) has a fast decay in q_i, then the convergence order of the Crank-Nicolson scheme can be improved to one for linear equations with additive noise , as in the case of SODEs.

For *spatial semi-discretization* methods for solving SPDEs (including but not limited to (3.1)), all methods can be applied, such as finite difference methods, finite element methods, finite volume methods, spectral methods, etc.

3.2. **Preprocessing methods for parabolic SPDEs.** In this type of methods, the underlying equation is first transformed into an equivalent form, which may bring some benefits in computation, and then is dealt with time discretization techniques. For example, splitting techniques split the underlying equation into a stochastic part and a deterministic part and save computational cost if either part can be efficiently solved either numerically or even analytically. In the splitting methods, we also have the freedom to use different schemes for different parts.

We will discuss two methods in this class: splitting techniques and exponential integrator methods.

3.2.1. *Splitting methods.* Splitting methods are also known as fractional step methods, and sometimes as predictor-corrector methods. They have been widely used for their computational convenience. Typically, the splitting is formulated by the following Lie-Trotter splitting, which splits the underlying problem, say (3.2), into two parts: 'stochastic part' (3.3a) and 'deterministic part' (3.3b). Consider the following Cauchy problem

$$du(t,x) = \mathcal{L}u(t,x)\,dt + \sum_{k=1}^{d_1} \mathcal{M}_k u(t,x) \circ dW_k, \quad (t,x) \in (0,T] \times \mathcal{D}, \tag{3.2}$$

where $\mathcal{L}$ is linear second-order differential operator, $\mathcal{M}_k$ is linear differential operator up to first order, and $\mathcal{D}$ is the whole space $\mathbb{R}^d$. The typical Lie-Trotter splitting scheme for (3.2) reads, over the time interval $(t_n, t_{n+1}]$, in integral form

$$\tilde{u}_n(t,x) = u_n(t_n,x) + \int_{t_n}^{t} \sum_{k=1}^{d_1} \mathcal{M}_k \tilde{u}_n(s,x) \circ dW_k(s), \quad t \in (t_n, t_{n+1}], \tag{3.3a}$$

$$u_n(t,x) = \tilde{u}_n(t_{n+1}) + \int_{t_n}^{t} \mathcal{L}u_n(s,x)\,ds, \quad t \in (t_n, t_{n+1}]. \tag{3.3b}$$

Here $\mathcal{M}_k$ can a zeroth-order or a first-order differential operator. We may expect half-order mean-square convergence.

3.2.2. *Integrating factor (exponential integrator) techniques.* In this approach, we first write the underlying SPDE in mild form (integration-factor) and then combine different time-discretization methods to derive fully discrete schemes.

In this approach, it is possible to derive high-order schemes in the strong sense since we may incorporate the dynamics of the underlying problems, as shown for ODEs with smooth random inputs. By formulating Equation (3.1) with additive noise ($g = 1$) in a mild form, we have

$$X(t) = e^{\mathcal{A}t} X_0 + \int_0^t e^{\mathcal{A}(t-s)} f(X(s))\,ds + \int_0^t e^{\mathcal{A}(t-s)}\,dW^Q(s), \tag{3.4}$$

then we can derive an exponential Euler scheme:

$$X_{k+1} = e^{\mathcal{A}h}[X_k + hf(X_k) + W^Q(t_{k+1}) - W^Q(t_k)], \tag{3.5}$$

or

$$X_{k+1} = e^{\mathcal{A}h} X_k + \mathcal{A}^{-1}(e^{\mathcal{A}h} - I) f(X_k) + \int_{t_k}^{t_{k+1}} e^{\mathcal{A}(t_{k+1}-s)}\,dW^Q(s), \tag{3.6}$$

where $t_k = kh$, $k = 0, \cdots, N$, $Nh = T$.

In certain cases, the total computational cost for the exponential Euler scheme can be reduced when $\eta_k = \int_{t_k}^{t_{k+1}} e^{\mathcal{A}(t_{k+1}-s)}\,dW^Q(s)$ is simulated as a whole instead of using increments of Brownian motion. For example, when $\mathcal{A}e_i = -\lambda_i e_i$, we observe that η_k solves the following equation

$$Y = \sum_{i=1}^{\infty} \int_{t_k}^{t_{k+1}} \mathcal{A} Y \, ds + \sum_{i=1}^{\infty} \int_{t_k}^{t_{k+1}} \sqrt{q_i} e_i \, dW_i(s), \tag{3.7}$$

and thus η_k can be represented by

$$\eta_k = \sum_{i=1}^{\infty} \sqrt{\gamma_i} e_i(x) \xi_{k,i}, \quad \xi_{k,i} = \frac{1}{\sqrt{\gamma_i}} \int_{t_k}^{t_{k+1}} e^{\lambda_i (t_{k+1}-s)} \, dW_i(s), \quad \gamma_i = \frac{q_i}{2\lambda_i}(1 - \exp(2\lambda_i h). \tag{3.8}$$

In this way, we incorporate the interaction between the dynamics and the noise, and thus we can have first-order mean-square convergence for additive noise.

Remark 3.1 (Weak convergence)**.** *Similar to the weak convergence of numerical methods for SODEs, the main tool for the weak convergence is the* Kolmogorov equation *associated with the functional and the underlying SPDE* [Da Prato, 2004, Da Prato and Zabczyk, 1992]. *For linear equations, the Kolomogrov equation for SPDEs is sufficient to obtain optimal weak convergence. A basic observation is that the weak convergence order is twice that of strong convergence.*

For nonlinear equations, Malliavin calculus *for SPDEs is usually applied for optimal weak convergence.*

Remark 3.2 (Pathwise convergence)**.** *There are two approaches to obtain pathwise convergence. The first is via mean-square convergence, using Borel-Cantelli lemma. The second approach is without knowing the mean-square convergence. In this approach, it was shown that it is crucial to establish pathwise regularity of the solution to obtain a pathwise convergence order.*

3.3. **Notes on numerical SPDEs.** For SPDEs driven by space-time noise, their solutions are usually of low regularity, especially when the noise is space-time white noise. Hence, it is difficult to obtain efficient high-order schemes.

Because of the low regularity of solutions to SPDEs, it is helpful to make full use of specific properties of the underlying SPDEs and preprocessing techniques to derive higher order schemes while keeping the computational cost low. For example, we can use the exponential Euler scheme (3.6) with (3.8) when the underlying SPDEs are driven by additive noise, and their leading differential operators are independent of randomness and time. When SPDEs (with multiplicative noises) have commutative noises, we can use the Milstein scheme (first-order strong convergence) while only sampling increments of Brownian motions.

Another issue for numerical methods of SPDEs is to reduce their computational cost in high-dimensional random space as there are usually infinite-dimensional stochastic processes whose truncations converge very slowly. This is the case even when high-order schemes like (3.8) can be used. Efficient infinite-dimensional integration methods should be employed to obtain the desired statistics with a reasonable computational cost. See Chapter 2 [Zhang and Karniadakis, 2017] for a brief review of numerical integration methods in random space.

SPDEs are usually solved with Monte Carlo methods, and many similar deterministic equations have to be solved. In some special cases, however, SPDEs can be solved in a very efficient way. For example, for some SPDEs with periodic boundary conditions, we can transform the equation into a deterministic one, which can then be solved once with deterministic solvers.

Appendix A. Facts for the proof of convergence orders

Recall that

$$(I-\nu\delta t\theta D_2)^{-1}(I+\nu\delta t(1-\theta)D_2) = \exp(\nu D_2\Delta t)+(\theta-\frac{1}{2})(\delta t\nu D_2)^2+(\theta^2-\frac{1}{6})(\delta t\nu D_2)^3+o((\delta t)^3).$$

and also

$$\int_{t_l}^{t_{l+1}} G_{t_{l+1}-s}\, dW^Q(s) = \sum_{k=1}^{\infty}\sqrt{q_k}e_k(x)\int_{t_l}^{t_{l+1}}\exp(-\lambda_k(t_{l+1}-s))\, dW_k(s) \sim \mathcal{N}(0, \sum_{k=1}^{\infty} q_k e_k(x)e_k(y)\frac{1-e^{-2\lambda_k\delta t}}{2\lambda_k}).$$

Denote $F_n(\nu\delta\Delta) =: \mathfrak{A}^n - \exp(\nu t_n D_2)$. Then $\|F_n(\nu\delta D_2)v\| \le C(\delta t)^q\, |v|_{2q}$ for $t_n \ge 0$, where

$$|v|_s = ((-D_2)^s v, v)^{1/2} = \left\|(-D_2)^{1/2}v\right\|_s = (\sum_{j=1}^{N}\lambda_j^s(v,\phi_j)^2)^{1/2}$$

Here λ_j are eigenvalues of $-D_2$ and ϕ_j are corresponding eigenfunctions.

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