LECTURE 9-10-A STOCHASTIC CALCULUS FOR FOR INFINITE DIMENSIONAL NOISE

Consider an infinite-dimensional Wiener process $W^Q(t)$ taking values in some Hilbert space H and with covariance operator Q.

Define the following cylindrical process on a bounded domain $\mathcal{D}$.

$$\dot{W}^Q(t,x) = \sum_{k=1}^{\infty} \sqrt{q_k} e_k(x) \dot{W}_k(t), \quad x \in \mathcal{D} \subsetneq \mathbb{R}^d. \tag{0.1}$$

Here $\{e_k(x)\}_k$ is a complete orthonormal basis in $L^2(\mathcal{D})$ and W_k's are i.i.d. standard Brownian motion. Here q_k's are the eigenvalues of the self-adjoint compact operator Q and $m_k(x)$ are the corresponding eigenfunctions (thus $\{e_k(x)\}_{k\geq 1}$ forms a complete orthonormal basis in $L^2(\mathcal{D})$). When $Q = I$, the process is called the space-time white noise.

Let $f = f(t)$ be an H-valued process such that, for every $h \in H$, the process $z(t) = (f(t), h)_H$ is adapted and

$$\mathbb{E} \int_0^T (\mathrm{Q} f(t), f(t))_H dt < \infty.$$

We call this process f admissible. Denote

$$\int_0^t f(s)\, dW^Q(s) = \int_0^t (f(s,y),\, dW^Q(s,y))_H \tag{0.2}$$

If $H = L^2(\mathcal{D})$,

$$\int_0^t f(s)\, dW^Q(s) = \int_0^t \int_{\mathcal{D}} f(s,y)\, dW^Q(s,y) = \sum_{k=1}^{\infty} \sqrt{q_k} \int_0^t f_k(s)\, dW_k(s). \tag{0.3}$$

where $f_k = (f, e_k)_{L^2(\mathcal{D})}$.

Let's discuss the properties of this integral.

- (Linearity) For real numbers a, b, we have

$$\begin{aligned} \int_0^t af(s) + bg(s)\, dW^Q(s) &= a\int_0^t f(s)\, dW^Q(s) + b\int_0^t g(s)\, dW^Q(s) \\ \int_0^t f(s)\, d(aW^Q(s) + bW^{Q_2}(s)) &= a\int_0^t f(s)\, dW^{Q_1}(s) + b\int_0^t f(s)\, dW^{Q_2}(s) \end{aligned}$$

- (isometry)

$$\mathbb{E}\left(\int_0^t f(s)\, dW^Q(s)\right)^2 = \sum_{k=1}^{\infty} q_k \int_0^t f_k^2(s)\, ds = \mathbb{E}\int_0^t (\mathrm{Q}(s)f(s), f(s))_{L^2(\mathcal{D})}\, ds.$$

- The integral $\int_0^t f(s)\, dW^Q(s)$ is a real-valued continuous square-integrable martingale.

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1. Stochastic integration with respect to H-valued martingales

When $\sum_{k=1}^{\infty} q_k$ is finite (or Q is a trace-class/nuclear operator), we can develop a nice theory for stochastic integration with respect to W^Q and H-valued martingale. In this section, we assume that Q is trace class.

Theorem 1.1. *Let X, Y be separable Hilbert spaces, and let $A : X \to Y$ be a bounded linear operator. Then A is nuclear if and only if*

(a) A is compact and $\sum_{k=1}^{\infty} \lambda_k < \infty$, where λ_k's are eigenvalues of $\sqrt{A^*A}$. (b) or A is trace class, i.e., $\sum_{k=1}^{\infty} (\sqrt{A^*A} m_k, m_k)_X < \infty$ for every orthonormal collection $\{m_k\}_{k\geq 1}$ in X.

Theorem 1.2 (Theorem 14.3, [Métivier and Pellaumail, 1980])**.** *Let M be a continuous square-integrable martingale with values in a separable Hilbert space H and $M(0) = 0$. Then*

(a) the process $\|M\|_H^2$ is a non-negative sub-martingale and there exists a unique continuous non-decreasing process $\langle M \rangle$ such that $\|M\|_H^2 - \langle M \rangle$ is a martingale.

b) There exists a unique process $Q_M(t)$ (Correlation operator of the martingale M) such that

- *for every t and ω, Q_M is a non-negative definite self-adjoint nuclear operator on H.*
- *for every $f, g \in H$,*

$$\langle (M, f)_H, (M, g)_H \rangle (t) = \int_0^t (\mathrm{Q}_M(s)f, g)_H \, d\langle M \rangle (s). \tag{1.1}$$

Exercise 1.3. *a) Compute the quadratic variation of W^Q when Q is a trace-class operator. (Hint. Show that W^Q is a $L^2(\mathcal{D})$-martingale; then find the quadratic variation is $t\mathrm{Tr}(Q)$).*

b) What is the correlation operator of W^Q?

Exercise 1.4. *Let $f_k = (f, e_k)_H$ where f is admissible. Define*

$$W_f^{\mathrm{Q}}(t) = \sum_{k\geq 1} \sqrt{q_k} \left(\int_0^t f_k(s) dW_k(s) \right) e_k.$$

Show that $W_f^{\mathrm{Q}}(t)$ is a real-valued continuous, square-integrable martingale and, for all admissible $f, g \in H$,

$$\langle W_f^{\mathrm{Q}}, W_g^{\mathrm{Q}} \rangle (t) = \frac{\langle W_f^{\mathrm{Q}} + W_g^{\mathrm{Q}} \rangle (t)}{4} - \frac{\langle W_f^{\mathrm{Q}} - W_g^{\mathrm{Q}} \rangle (t)}{4} = \int_0^t (\mathrm{Q}f(s), g(s))_H ds. \tag{1.2}$$

Theorem 1.5 (Burkholder-Davis-Gundy (BDG) Inequality))**.** *If H is a separable Hilbert space and M is an H-valued continuous square-integrable martingale with $M(0) = 0$, then, for every $p > 0$, there exists a positive number C, depending only on p, such that*

$$\mathbb{E} \left(\sup_{t\leq T} \|M(t)\|_H \right)^p \leq C\mathbb{E}(\langle M \rangle (T))^{p/2}. \tag{1.3}$$

The proof is based on Monotone Convergence Theorem and BDG in finite dimension.

1.1. Stochastic integration.

Definition 1.6 (M-admissible integrand)**.** *An H-valued process $f = f(t)$ is called an M-admissible integrand if, for every $h \in H$, the process $x(t) = (f(t), h)_H$ is a continuous adapted process and*

$$\mathbb{E} \int_0^t (\mathrm{Q}_M(s) f(s), f(s))_H \, d\langle M \rangle (s).$$

where Q_M is the correlation operator of M.

Definition 1.7. *Given an orthonormal basis* $\{m_k\}_{k\geq 1}$ *in* H *and an* M*-admissible integrand* f*, we define*

$$M_f(t) = \int_0^t (f(s), dM(s))_H = \sum_{k\geq 1} \int_0^t f_k(s)\, dM_k,$$

where $f_k(t) = (f(t), m_k)_H$ *and* $M_k(t) = (M(t), m_k)_H$.

The definition of M_f does not depend on the choice of the basis in H. The fact can be proved by computing the cross quadratic variation and applying the BDG inequality.

Properties

- (Linearity) For real numbers a, b, we have

$$(aM + bN)_f = aM_f + bN_f, \quad M_{af+bg} = aM_f + bM_g. \tag{1.4}$$

- (isometry)

$$\mathbb{E}\left(\int_0^t (f(s), dM(s))_H\right)^2 = \mathbb{E}\int_0^t (\mathrm{Q}_M(s)f(s), f(s))_H\, d\langle M\rangle(s). \tag{1.5}$$

- M_f is a real-valued continuous square-integrable martingale and

$$\langle M_f\rangle(t) = \int_0^t (\mathrm{Q}_M(s)f(s), f(s))_H\, d\langle M\rangle(s). \tag{1.6}$$

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$$\mathbb{E}[\sup_{0<t\leq T}\left|\int_0^t (f(s), dM(s))_H\right|] \quad \leq C\mathbb{E}\left(\int_0^T (\mathrm{Q}_M(s)f(s), f(s))_H\, d(M)(s)\right)^{1/2}, \tag{1.7}$$

If $\mathbb{E}\sup_{0<t\leq T}\|f(s)\|_H^2 < \infty$, then

$$\mathbb{E}[\sup_{0<t\leq T}\left|\int_0^t (f(s), dM(s))_H\right|] \leq \frac{1}{2}\mathbb{E}[\sup_{0<t\leq T}\|f(s)\|_H^2] + C_1\mathbb{E}[\langle M\rangle](T).$$

Exercise 1.8. *Show that*

$$\langle M_f, M_g\rangle(t) = \int_0^t (\mathrm{Q}_M(s)f(s), g(s))_H\, d\langle M\rangle(s)$$

1.2. **Ito's formula in infinite dimension.**

Theorem 1.9 (Infinite-Dimensional Itô Formula)**.** *Let* H *be a separable Hilbert space. Consider the real-valued function* $F = F(t, h)$*,* $t \in [0, T]$*,* $h \in H$*, and an* H*-valued process* $X = X(t)$ *and assume that*

- F*,* F_t*,* DF*,*D^2F *exist and are continuous in* $[0, T]\times H$*, and are bounded and uniformly continuous on bounded sub-sets of* $[0, T] \times H$*.*
- *There exists an* H*-valued adapted process* B *with* $\mathbb{E}[\int_0^T \|B(t)\|_H^2\, dt]$ *and an* H*-valued continuous, square-integrable martingale* M *with correlation operator* Q_M *and* $M(0) = 0$*, such that*

$$X(t) = X(0) + \int_0^T B(s)ds + M(t). \tag{1.8}$$

Then

$$\begin{aligned} F(t, X(t)) &= F(0, X(0)) + \int_0^t F_t(s, X(s))ds + \int_0^t (DF(s, X(s)), dX(s)) \\ &\quad + \frac{1}{2}\int_0^t \mathrm{tr}\left(D^2F(s, X(s))\mathrm{Q}_M(s)\right) d\langle M\rangle(s) \end{aligned} \tag{1.9}$$

Exercise 1.10. *Verify that*

$$\|X(t)\|_H^2 = \|X(0)\|_H^2 + 2\int_0^t (X(s), B(s))_H ds + 2\int_0^t (X(s), dM(s))_H + \langle M\rangle(t). \tag{1.10}$$

1.3. **Generalization.** A collection of random operators $B(t)$ from H to X is called an M-admissible integrand if

- For every $h \in H$ and $x \in X$, the process $g(t) = (B(t)h, x)_X$ is continuous and adapted.
- For every $t \in (0, T]$ and $\omega \in \Omega$, the operator $B(t,\omega)Q_M(t,\omega)B^*(t,\omega)$ (from X to X) is trace-class and

$$\mathbb{E}\int_0^T \mathrm{tr}\left(\mathrm{B}(t)\mathrm{Q}_M(t)\mathrm{B}^*(t)\right)d\langle M\rangle(t) < \infty. \tag{1.11}$$

Definition 1.11. *Let H, X be separable Hilbert spaces and $M = M(t)$ be a continuous H-valued square-integrable martingale. Assume that $M(0) = 0$ and $B : H \to X$ is an admissible integrand. We define the stochastic integral*

$$M_B(t) = \int_0^t B(s)\, dM(s),$$

as the unique X-valued process such that for all $x \in X$, $t \in [0, T]$,

$$(M_B(t), x)_X = \int_0^t (B^*(s)x, dM(s))_H. \tag{1.12}$$

Here we only need to show the process $f(t) = B^*(s)x$ is an M-admissible integrand.

Theorem 1.12. *The process M_B is an X-valued continuous, square-integrable martingale and for all $x, y \in X$,*

$$\langle (M_B(t), x)_X, (M_B(t), y)_X\rangle(t) = \int_0^t (\mathrm{B}(s)\mathrm{Q}_M(s)\mathrm{B}^*(s)x, y)\, d\langle M\rangle(s). \tag{1.13}$$

Choosing an orthonormal basis in X and we can also show that

$$\mathbb{E}[\|M_B(t)\|_X^2] = \mathbb{E}[\int_0^t (\mathrm{B}(s)\mathrm{Q}_M(s)\mathrm{B}^*(s)x, y)\, d\langle M\rangle(s)] \tag{1.14}$$

Theorem 1.13 (Burkholder-Davis-Gundy (BDG) Inequality)**.**

$$\mathbb{E}\sup_{0<t\le T}\left\|\int_0^t B(s)dM(s)\right\|^p \quad \le C_p\mathbb{E}\left(\int_0^T \left(\mathrm{tr}\left(\mathrm{B}(s)\mathrm{Q}_M(s)\mathrm{B}^*(s)\right)\right) d\langle M\rangle(s)\right)^{p/2}. \tag{1.15}$$

References

[Métivier and Pellaumail, 1980] Métivier, M. and Pellaumail, J. (1980). *Stochastic integration.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London-Toronto, Ont. Probability and Mathematical Statistics.