LECTURE 8 STOCHASTIC PARABOLIC EQUATIONS AND THEIR NUMERICAL METHODS: I. FINITE DIMENSIONAL NOISE

1. Mean-square stability of stochastic advection-diffusion equations with single noise

We can define linear stability, which is concerned the with asymptotic behavior of numerical solutions when the time step K goes to infinity while the time step size δt is fixed. The linear stability for evolutionary SPDEs is a straightforward extension of linear stability for SODEs, which can also be defined in the mean-square, almost sure or weak sense. The concern of linear stability often leads to the Courant-Friedrichs-Lewy condition (often abbreviated as CFL condition): the mesh size in time has to be proportional to a certain power (depending on the order of leading operator) of the mesh size in space. This is similar to linear stability of PDEs.

1.1. **Additive noise.** Consider the following problem with the vanishing Dirichlet boundary condition.

$$dv = \mu\partial_x v dt + \nu\partial_x^2 v dt + \sigma dW(t), \quad \nu > 0,\, x \in (0,1) \tag{1.1}$$

Denote $V^{k+\theta} = \theta V^{k+1} + (1-\theta)V^k$, $\theta \in [0,1]$. The one-step θ-scheme is

$$V^{k+1} = V^k + \delta t(\mu D_1 + \nu D_2)V^{k+\theta} + \sigma\sqrt{\delta t}\xi_k, \tag{1.2}$$

where $\Delta W_n = W(t_{k+1}) - W(t_k) \equiv \xi_k\sqrt{\delta t}$ and D_1 and D_2 are the first and second central difference operators in x. Here ξ_k are i.i.d. independent standard Gaussian random variables.

By induction, we obtain

$$V^{k+1} \;=\; \mathfrak{A}^{k+1}V^0 + \sqrt{\delta t}\sum_{j=1}^{k}\mathfrak{A}^j\mathfrak{B}\mathbf{1}\xi_j \tag{1.3}$$

where $\mathfrak{A}$ is the (amplification factor) and $\mathfrak{B}$ are defined as follows.

$$\mathfrak{A} \;=\; (I - \delta t\theta(\mu D_1 + \nu D_2))^{-1}(I + \delta t(1-\theta)\theta(\mu D_1 + \nu D_2))^{-1}. \tag{1.4}$$

$$\mathfrak{B} \;=\; \sigma(I - \delta t\theta(\mu D_1 + \nu D_2))^{-1}. \tag{1.5}$$

Taking the expectation over both sides of (1.3), we obtain

$$\mathbb{E}[V^{k+1}] = \mathfrak{A}\mathbb{E}[V^k] = \mathfrak{A}^{k+1}\mathbb{E}[V^0].$$

Then the stability condition in the mean will be the same as that for the case $\sigma = 0$ (non-random equation). Here we need the eigenvalues of $\mathfrak{A}$ to be less than 1. Then we need $\frac{2}{2\theta-1} < \lambda\delta t < 0$ if $0 \le \theta < 1/2$ and $\lambda h < 0$ if $\frac{1}{2} \le \theta \le 1$, where λ is the largest eigenvalue (in the absolute value) of the matrix $\mu D_1 + \nu D_2$. When $\mu = 0$, we can readily derive the stability condition as $\lambda = -2/h^2(1 + \cos(\pi/(n-1)))$.

For the mean-square stability, we have various metrics to choose in x. We may take any positive-definite (even non-negative definite) matrix M to measure the stability of $(V^{k+1})^\top MV^{k+1}$. For example, taking $M = I$ gives the discrete analogue of the L^2-norm and thus leads to the l^2-stability; taking $M = D_2$ gives a discrete analogue of the H^1-seminorm.

$$\mathbb{E}[(V^{k+1})^\top MV^{k+1}] \;=\; \mathbb{E}[(V^k)^\top M\mathfrak{A}V^k] + \sigma^2\delta t\mathbb{E}[(\mathfrak{B}\mathbf{1})^\top M\mathfrak{B}\mathbf{1}].$$

Date: November 5, 2019.

From here we can derive stability conditions.

Absolute stability (for certain θ) can be also derived by requiring the numerical solution to have the same asymptotic covariance matrix as the analytical solution.

Remark 1.1. *Here the roles of μ and ν can be significant in determining λ (of the matrix $\mu D_1 + \nu D_2$). When μ is large while ν is small, we may have a convection-dominate phenomenon. Then the dominant eigenvalue would be from advection not diffusion. In physics, it is the ratio (Peclet number) $\mu/\nu \times L$ (L is the characteristic length scale; here $L = 1$) that expresses the relative importance of advection compared to diffusion.*

The stability analysis above is known as *the matrix method*. This is the most general method and is suitable for studying the effect of boundary conditions on the stability of the difference equation.

1.2. **Multiplicative noise.** Let's consider the linear stability of the θ-scheme for

$$dv = \mu \partial_x v dt + \nu \partial_x^2 v dt + \sigma \partial_x u dW(t), \quad x \in (0,1) \tag{1.6}$$

with the periodic boundary condition.

$$V^{k+1} = V^k + (\mu D_1 V^{k+\theta} + \nu D_2 V^{k+\theta})\delta t + \sigma \sqrt{\delta t} D_1 V^n \xi_k, \tag{1.7}$$

For the mean-square scheme, we obtain

$$\mathbb{E}[(V^{k+1})^2] = \mathbb{E}[(\mathfrak{A} V^k)^2] + \sigma^2 \delta t \mathbb{E}[\mathfrak{B} D_1 V^k)^2] \tag{1.8}$$

or

$$\mathbb{E}[(V^{k+1})^\top M V^{k+1}] \;=\; \mathbb{E}[(V^k)^\top M \mathfrak{A} V^k] + \sigma^2 \delta t \mathbb{E}[(V^k)^\top (\mathfrak{B} D_1)^\top M \mathfrak{B} D_1 V^k].$$

We then apply the matrix method to obtain proper stability conditions (left as an exercise). Intuitively, we cannot obtain better stability conditions for multiplicative noise than that for additive noise.

1.2.1. *Improved stability condition.* Let's consider the linear stability of the first-order σ-θ-scheme. Consider the following problem with periodic (or vanishing Dirichlet) boundary conditions:

$$dv = \mu \partial_x v dt + \nu \partial_x^2 v dt + \sigma \partial_x v dW(t), \quad 0 \le \sigma < \nu^2/2, x \in (0,1) \tag{1.9}$$

the σ-θ scheme reads, with time step size δt and space step size δx,

$$\begin{aligned} V^{k+1} \;=\; & V^k + \theta \delta t (\mu D_1 + \nu D_2) V^{k+1} + (1-\theta)\delta t (\mu D_1 + \nu D_2) V^n \quad \text{(`deterministic part')} \\ & -\frac{\delta t}{2}\sigma^2 [\alpha D_2 V^{k+1} + (1-\alpha) D_2 V^k] \quad \text{(`correction term due to stochastic part')} \\ & + \sigma \sqrt{\delta t} D_1 V^n \xi_k + \frac{\sigma^2}{2} \delta t D_2 V^n \xi_k^2, \quad \text{(`stochastic part')} \end{aligned} \tag{1.10}$$

where ξ_k are i.i.d. independent standard Gaussian random variables, $\theta \in [0,1]$

Let

$$\begin{aligned} A &= I - \theta \delta t (\mu D_1 + \nu D_2) + \frac{\delta t}{2} \sigma^2 \alpha D_2 \\ B &= I + (1-\theta)\delta t (\mu D_1 + \nu D_2) - \frac{\delta t}{2} \sigma^2 (1-\alpha) D_2 \\ C &= \sigma \sqrt{\delta t} D_1 \xi_k + \frac{\sigma^2}{2} \delta t D_2 \xi_k^2 \end{aligned}$$

Then we have

$$V^{k+1} = A^{-1}BV^k + A^{-1}CV^k. \tag{1.11}$$

Exercise 1.2. *Apply the matrix method to obtain the stability condition for the scheme* (1.11) *when* $\mu = 0$.

Exercise 1.3. *Apply the matrix method of stability analysis for the scheme* (1.15) *when the vanishing Dirichlet boundary condition is imposed.*

Exercise 1.4. *Analyze the mean-square stability of the* θ*-scheme for*

$$dv = \mu\partial_x v dt + \nu\partial_x^2 v dt + \sigma u dW(t), \quad x \in (0,1) \tag{1.12}$$

with a vanishing Dirichlet boundary condition.

1.2.2. *The von Neumann Stability Analysis (Fourier analysis).* This approach was developed by John von Neumann in the early 1940s for PDEs and is probably the most popular method on periodic domain or the whole space. It assumes the discrete analog of Fourier series expansions in the form $V_j^k = \sum_{l=1}^{N-1} a_l^k(\omega)\exp(l\pi x_j)$ (or use other proper Fourier basis with boundary conditions satisfied). Here $V_0^k = V_N^k = 0$ and $x_j = j\delta x$.

The main idea is to reduce the difference equation for V_j^n into an uncoupled ordinary difference equation for a_l^k, and subsequently require that $\left|a_l^{k+1}\right| \le \left|a_l^k\right|$ in the mean, mean-square or pathwise sense.

1.2.3. *Light Version of the von Neumann Stability Analysis.* The light version of the Fourier stability analysis is to apply $a_l^k(\omega)\exp(il\pi x_j)$ for all l, $0 \le l \le N-1$ instead of a summation of them.

Now we assume that $\mu = 0$. Recall the scheme

$$\begin{aligned} V_j^{k+1} &= V_j^k + \nu\frac{V_{j+1}^{k+\theta} - 2V_j^{k+\theta} + V_{j-1}^{k+\theta}}{(\delta x)^2}\delta t + \sigma\frac{V_{j+1}^k - V_{j-1}^k}{2\delta x}\xi_k\sqrt{\delta} \\ &\quad -\frac{\sigma^2}{2}\frac{V_{j+1}^{k+\alpha} - 2V_j^{k+\alpha} + V_{j-1}^{k+\alpha}}{(\delta x)^2}\delta t + \frac{\sigma^2}{2}\frac{V_{j+1}^k - 2V_j^k + V_{j-1}^k}{(\delta x)^2}\delta t. \end{aligned}$$

Note that

$$V_{j+1}^{k+\theta} - 2V_j^{k+\theta} + V_{j-1}^{k+\theta} = a_l^{k+\theta}\exp(l\pi x_j)(\exp(i\phi) - 2 + \exp(-i\phi)) = -a_l^{k+\theta}\exp(l\pi x_j)4\sin^2(\frac{\phi}{2})$$

$$V_{j+1}^k - V_{j-1}^k = a_l^k\exp(l\pi x_j)2\sqrt{-1}\sin(\phi)$$

Plugging these in (1.10), we have

$$a_l^{k+1} = a_l^k + \nu\mathsf{a}Ra_l^{k+\theta} + \mathsf{c}\xi_k\sqrt{R}a_l^k - \frac{\sigma^2}{2}\mathsf{a}Ra_l^{k+\alpha} + \frac{\sigma^2}{2}\mathsf{a}R\xi_k^2 a_l^k.$$

where $\mathsf{a} = -4\sin^2(\frac{\phi}{2})$, $\mathsf{c} = \sqrt{-1}\sin(\phi)$ and $R = \frac{\delta t}{(\delta x)^2}$. Then we have

$$(1 - \mathsf{a}(\theta\nu - \alpha\frac{\sigma^2}{2})R)^2\mathbb{E}[\left|a_l^{k+1}\right|^2] = [(1 + \mathsf{a}((1-\theta)\nu + \alpha\frac{\sigma^2}{2})R)^2 + \frac{\sigma^4}{2}\mathsf{a}^2R^2 + \mathsf{c}^2R]\mathbb{E}[\left|a_l^k\right|^2]. \tag{1.13}$$

We then require that

$$(1 + \mathsf{a}((1-\theta)\nu + \alpha\frac{\sigma^2}{2})R)^2 + \frac{\sigma^4}{2}\mathsf{a}^2R^2 + \mathsf{c}^2R \le (1 - \mathsf{a}(\theta\nu - \alpha\frac{\sigma^2}{2})R)^2.$$

which can be simplified as

$$4R\sin^2(\frac{\phi^2}{2})\big(\nu(1-2\theta) + \alpha\sigma^2 + \frac{\sigma^4}{2\nu}\big) + \frac{\sigma^2}{2\nu}2\cos^2(\frac{\phi}{2}) < 2. \tag{1.14}$$

When $2R\big(\nu(1-2\theta)+\alpha\sigma^2+\frac{\sigma^4}{2\nu}\big)<1$ or

$$(2\nu)\frac{\delta t}{\delta x^2}[1-2(\theta-\frac{\sigma^2}{2\nu}\alpha-\frac{\sigma^4}{4\nu^2})]<1, \tag{1.15}$$

it holds that $\mathbb{E}[\left|a_l^{k+1}\right|^2]\leq\mathbb{E}[\left|a_l^k\right|^2]$.

when $\alpha=-1$, $\theta>1/2$ the scheme is unconditionally stable as we have, When $\alpha=0$, $\theta=0$, the scheme becomes the Milstein discretization in time. In Table 1, we summarize the CFL conditions for Equation (1.9) (when $\mu=0$) with various σ and different discretization parameters θ and α.

Remark 1.5. *When $\sigma=0$, the von Neumann necessary condition for stability is $\rho(A^{-1}B)\leq 1$ (can be relaxed to $1+O(\delta t)$) where $\rho(\cdot)$ is the spectral radius of the matrix '$\cdot$'. If $A^{-1}B$ (the amplification matrix) is normal (it commutes with its Hermitian conjugate), then the condition $\rho(A^{-1}B)\leq 1$ is also a sufficient condition.*

When $\sigma\neq 0$, the equivalence would require $A^{-1}B+A^{-1}C$ to be normal.

Exercise 1.6. *Is $A^{-1}B+A^{-1}C$ normal?*

TABLE 1. Stability region of the scheme (1.10) for Equation (1.9) when $\mu=0$.

σ	θ	α	CFL condition	scheme
0	0	–	$2\nu\frac{\delta t}{\delta x^2}<1$	explicit
0	$(0,\frac{1}{2})$	–	$2\nu\frac{\delta t}{\delta x^2}(1-2\theta)<1$	implicit
0	$[\frac{1}{2},1]$	–	–	implicit
$(0,\sqrt{2\nu})$	0	0	$\frac{\delta t}{\delta x^2}(1+\frac{\sigma^4}{2\nu^2})<1$	explicit
$(0,\sqrt{2\nu})$	$(0,\min(\frac{1}{2}+\frac{\sigma^4}{4\nu^2},1))$	0	$\frac{\delta t}{\delta x^2}[1-2(\theta-\frac{\sigma^4}{4\nu^2})]<1$	implicit
$(0,\sqrt{\nu}]$	$(\min(\frac{1}{2}+\frac{\sigma^4}{4\nu^2},1),1)$	0	–	implicit
$(0,\sqrt{2\nu})$	$[\frac{1}{2}-\frac{\sigma^2}{2\nu}+\frac{\sigma^4}{4\nu^2},1]$	-1	–	implicit
$[\sqrt{2\nu},\infty)$	–	–	–	Not mean-square stable

2. LINEAR STOCHASTIC ADVECTION-DIFFUSION-REACTION EQUATIONS

Consider the following SPDE written in Itô's form:

$$\begin{aligned} du(t,x) &= [\mathcal{L}u(t,x)+f(x)]\,dt+\sum_{k\geq 1}[\mathcal{M}_k u(t,x)+g_k(x)]\,dW_k(t),\ (t,x)\in(0,T]\times\mathcal{D}, \\ u(0,x) &= u_0(x),\quad x\in\mathcal{D}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \mathcal{L}u(t,x) &= \sum_{i,j=1}^d a_{ij}(x)\,D_iD_ju(t,x)+\sum_{i=1}^d b_i(x)D_iu(t,x)+c(x)\,u(t,x), \\ \mathcal{M}_k u(t,x) &= \sum_{i=1}^d b_i^k(x)D_iu(t,x)+h^k(x)\,u(t,x), \end{aligned} \tag{2.2}$$

and $D_i:=\partial_{x_i}$ and $\mathcal{D}$ be an open domain in $\mathbb{R}^d$. We assume that $\mathcal{D}$ is either bounded with a regular boundary or that $\mathcal{D}=\mathbb{R}^d$. In the former case we will consider periodic boundary conditions and in the latter the Cauchy problem. Let $(W(t),\mathcal{F}_t)=(\{W_k(t),k\geq 1\},\mathcal{F}_t)$

be a system of one-dimensional independent standard Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{F}_t,\ 0 \le t \le T$, is a filtration satisfying the usual hypotheses.

Remark 2.1. *The problem* (2.1)-(2.2) *can be regarded as a problem driven by a cylindrical Wiener process. Consider a cylindrical Wiener process*

$$W^Q(t,x) = \sum_{k=1}^{\infty} \sqrt{q_k} W_k(t) e_k(x), \tag{2.3}$$

where $\sum_{k=1}^{\infty} q_k < \infty$ $(q_k > 0)$ *and* $\{W_k(t)\}$ *are independent Wiener processes, and* $\{e_k(x)\}_{k=1}^{\infty}$ *is a complete orthonormal system (CONS) in* $L^2(\mathcal{D})$*. Thus, we can view* (2.1)-(2.2) *as SPDEs driven by this cylindrical Wiener process when* $\mathcal{M}_k u = e_k(x)\mathcal{M}u$ *and* $\mathcal{M}$ *is first-order or zeroth order differential operator.*

3. Existence and uniqueness

We assume the *coercivity condition* that there exist a constant $\delta_{\mathcal{L}} > 0$ and a real number $C_{\mathcal{L}}$ such that for any $v \in H^1(\mathcal{D})$,

$$\langle \mathcal{L}v, v\rangle + \frac{1}{2}\sum_{k\ge1} \|\mathcal{M}_k v\|^2 + \delta_{\mathcal{L}} \|v\|_{H^1}^2 \le C_{\mathcal{L}} \|v\|^2 , \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ is the duality between the Sobolev spaces $H^{-1}(\mathcal{D})$ and $H_0^1(\mathcal{D})$ associated with the inner-product over $L^2(\mathcal{D})$ and $\|\cdot\|$ is the $L^2(\mathcal{D})$-norm. A necessary condition for (3.1) is that the coefficients satisfy

$$\sum_{i,j=1}^{d} \left(2a_{i,j}(x) - \sum_{k\ge1} \sigma_{i,k}(x)\sigma_{k,j}(x) \right) y_i y_j \ge 2\delta_{\mathcal{L}} |y|^2 , \quad x, y \in \mathcal{D}.$$

With these assumptions, we have a unique square-integrable (variational) solution of (2.1)-(2.2) if we also have the following conditions:

- the coefficients of operators $\mathcal{L}$ and $\mathcal{M}$ in (2.2) are uniformly bounded and predictable for every $x \in \mathcal{D}$. The coefficients $a_{i,j}(x)$'s are Lipschitz continuous;
- For $\phi \in H^1(\mathcal{D})$, $\sum_{k\ge1} \mathbb{E}[\|\mathcal{M}_k \phi(t)\|_{L^2}^2] < \infty$;
- the initial condition $u_0(x) \in \mathbb{L}^2(\Omega; L^2)$ is $\mathcal{F}_0$-measurable;
- $f(t,\omega)$ and $g_k(t,\omega)$ are adapted and $\int_0^T \|f(t)\|_{H^{-1}}^2\, dt < \infty$, $\sum_{k\ge1} \int_0^T \|g_k(t)\|_{L^2}^2\, dt < \infty$.

Then for each $\phi \in H^1$ or a dense subset of H^1 and all $t \in [0,T]$, the adapted process $u(t)$ is a variational solution to (2.1)-(2.2). With the coercivity condition and $\|\mathcal{L}\phi\|_{H^{-1}} \le C_0 \|\phi\|_{H^1}$, there exists a unique solution $u \in \mathbb{L}^2(\Omega, C((0,T)); L^2(\mathcal{D})$ and satisfies

$$\mathbb{E}[\sup_{0<t<T} \|u(t)\|_{L^2}^2] + \frac{\delta_L}{2}\mathbb{E}[\int_0^T \|u(t)\|_V^2\, dt] \le C\mathbb{E}[\|u_0\|_{L^2}^2] + C\mathbb{E}[\int_0^T \|f(t)\|_{H^{-1}}^2\, dt] + C\sum_{k\ge1}\int_0^T \|g_k(t)\|_{L^2}^2\, dt.$$

Here C depends on C_0, $C_{\mathcal{L}}$, $\delta_{\mathcal{L}}$ and T.

The existence and uniqueness of the solution is a special case of the following conclusion.

Theorem 3.1 (Energy Equality in Hilbert Spaces, [Lototsky and Rozovsky, 2017])**.** *Let* (V, H, V') *be a normal triple of Hilbert spaces and* $\langle \cdot, \cdot \rangle$ *the duality between* V *and* V' *relative* *to the inner product in H. Let x be an F_0-measurable random element, and let $X = X(t)$, $B = B(t)$, and $M = M(t)$ be adapted processes with the following properties:*

- *x is H-valued and $\mathbb{E}[\|x\|_H]^2 < \infty$;*
- *x is V-valued and $\mathbb{E}[\int_0^T \|X(t)\|_V]^2\, dt < \infty$;*
- *B is V'-valued and $\mathbb{E}[\int_0^T \|B(t)\|_{V'}]^2\, dt < \infty$;*
- *M is H-valued continuous, square-integrable martingale;*
- *For every $v \in V$ and every $t \in [0, T]$, it holds with probability one that*

$$(X(t), v)_H = (x, v)_H + \int_0^t [B(s), v] ds + (M(t), v)_H \tag{3.2}$$

Then it holds for all $t \in [0, T]$.

$$\|X(t)\|_H^2 = \|x\|_H^2 + 2\int_0^t [B(s), X(s)] ds + 2\int_0^t (X(s), dM(s))_H + \langle M(t)\rangle. \tag{3.3}$$

Moreover, X has continuous trajectories as an H-valued process, and there is a number C, independent of T, such that

$$\mathbb{E}\sup_{0<t<T} \|X(t)\|_H^2 \le C\left(\mathbb{E}\|x\|_H^2 + \mathbb{E}\int_0^T \|X(t)\|_V^2 dt + \mathbb{E}\int_0^T \|B(t)\|_{V'}^2 dt + \mathbb{E}\langle M\rangle(T)\right). \tag{3.4}$$

Here the last inequality follows from the Burkholder-Davis-Gundy inequality and energy equality.

3.1. **Conversion between Ito and Stratonovich formulation.** In Stratonovich form, (2.1)-(2.2) is written as

$$\begin{aligned} du(t,x) &= [\tilde{\mathcal{L}}u(t,x) + f(x)]\, dt + \sum_{k=1}^{q} [\mathcal{M}_k u(t,x) + g_k(x)] \circ \dot{W}_k\, dt,\ (t,x) \in (0,T] \times \mathcal{D}, \\ u(0,x) &= u_0(x), \quad x \in \mathcal{D}, \end{aligned} \tag{3.5}$$

where $\tilde{\mathcal{L}}u = \mathcal{L}u - \frac{1}{2}\sum_{1\le k\le q} \mathcal{M}_k[\mathcal{M}_k u + g_k]$.

Example 3.2. *Consider the following one-dimensional equation for $(t, x) \in (0, T] \times (0, 2\pi)$:*

$$du = [(\epsilon + \frac{1}{2}\sigma^2)\partial_x^2 u + \beta \sin(x)\partial_x u]\, dt + \sigma\partial_x u\, dW(t), \tag{3.6}$$

where $W(t)$ is a standard scalar Brownian motion (Wiener process), $\epsilon > 0$, β, σ are constants.

In the Stratonovich form, Equation (3.6) can be written as

$$du = [\epsilon\partial_x^2 u + \beta \sin(x)\partial_x u]\, dt + \sigma\partial_x u \circ dW(t). \tag{3.7}$$

We may consider the fully implicit midpoint scheme for the (3.6) in Stratonovich form: (when $f = g_r = 0$)

$$u^{n+1} = u^n + \delta t\tilde{\mathcal{L}}u^{n+1/2}\, dt + \sum_{r=1}^{q} \mathcal{M}_k u^{n+1/2}\zeta_{r,k}\sqrt{\delta t}, \tag{3.8}$$

where $\zeta_{r,k}$ is a truncation of the standard normal random vector $\xi_{r,k}$:

$$\zeta_{r,k} = \xi_{r,k}\chi_{[-M_{\delta t}, M_{\delta t}]} + M_{\delta t}\xi_{r,k}\chi_{(M_{\delta t},\infty)} - M_{\delta t}\xi_{r,k}\chi_{(-\infty, M_{\delta t})}. \tag{3.9}$$

Here $M_{\delta t} = \sqrt{2l\,|\ln(\delta t)|}$, $l \ge 1$. It can be clearly shown that

$$.0 < \mathbb{E}[\xi_{r,k}^2 - \zeta_{r,k}^2] < (1 + 2l|\ln \delta t|)(\delta t)^l. \tag{3.10}$$

One advantage of this midpoint scheme is that it converges at order 1 in the mean-square sense when $q = 1$, which is a special case of commutative noises.

Exercise 3.3. *Prove the estimate* (3.10)*.*

3.2. **Commutative noises.** The problem (2.1)-(2.2) is said to have *commutative noises* if

$$\mathcal{M}_k\mathcal{M}_j = \mathcal{M}_j\mathcal{M}_k, \quad 1 \le k, j \le q, \tag{3.11}$$

and to have non-commutative noises otherwise. When $q = 1$, (3.11) is satisfied and thus this is a special case of commutative noises. When $\mathcal{M}_k$ are zeroth-order operators, ($\sigma_{i,k} = 0$), (3.11) is satisfied and the problem also has commutative noises. The definition is consistent with that of commutative and non-commutative noises for stochastic ordinary differential equations.

Example 3.4. *Consider the following one-dimensional equation for* $(t, x) \in (0, T] \times (0, 2\pi)$*:*

$$\begin{aligned} du &= [(\epsilon + \frac{1}{2}\sigma_1^2 \cos^2(x))\partial_x^2 u + (\beta \sin(x) - \frac{1}{4}\sigma_1^2 \sin(2x))\partial_x u]\, dt \\ &\quad + \sigma_1 \cos(x)\partial_x u\, dW_1(t) + \sigma_2 u\, dW_2(t), \end{aligned} \tag{3.12}$$

where $(W_1(t), W_2(t))$ *is a standard two-dimensional Wiener process,* $\epsilon > 0$*,* β*,* σ_1*,* σ_2 *are constants.*

In the Stratonovich form, Equation (3.12) is written as

$$du = [\epsilon\partial_x^2 u + \beta \sin(x)\partial_x u]\, dt + \sigma_1 \cos(x)\partial_x u \circ dW_1(t) + \sigma_2 u \circ dW_2(t). \tag{3.13}$$

The problem has commutative noises (3.11):

$$\sigma_1 \cos(x)\partial_x \sigma_2 \mathrm{Id} u = \sigma_2 \mathrm{Id} \sigma_1 \cos(x)\partial_x = \sigma_1\sigma_2 \cos(x)\partial_x.$$

Here Id is the identity operator.

Example 3.5. *Consider the following one-dimensional equation for* $(t, x) \in (0, T] \times (0, 2\pi)$*:*

$$\begin{aligned} du &= [(\epsilon + \frac{1}{2}\sigma_1^2)\partial_x^2 u + \beta \sin(x)\partial_x u + \frac{1}{2}\sigma_2^2 \cos^2(x) u]\, dt \\ &\quad + \sigma_1 \partial_x u\, dW_1(t) + \sigma_2 \cos(x) u\, dW_2(t), \end{aligned} \tag{3.14}$$

where $(W_1(t), W_2(t))$ *is a standard Wiener process,* $\epsilon > 0$*,* β*,* σ_1*,* σ_2 *are constants.*

In the Stratonovich form, Equation (3.14) is written as

$$du = [\epsilon\partial_x^2 u + \beta \sin(x)\partial_x u]\, dt + \sigma_1 \partial_x u \circ dW_1(t) + \sigma_2 \cos(x) u \circ dW_2(t). \tag{3.15}$$

The problem has non-commutative noises as the coefficients do not satisfy (3.11).

4. Numerical methods for SPDEs

Let us introduce the stability and convergence of numerical schemes. We denote $\delta t_k = (t_{k+1} - t_k)$ ($k = 1, 2, \ldots, K$, $\sum_{k=1}^K \delta t_k = T$) are the time step sizes. Sometimes we simply use the time step size δt when all δt_k's are equal. We also denote by $N > 0$ the number of orthogonal modes in spectral methods or discretization steps in space ($Nh = |\mathcal{D}|$, $|\mathcal{D}|$ is the length of the interval $\mathcal{D} \subset \mathbb{R}^d$ when $d = 1$) for finite difference methods or finite element methods. We denote a numerical solution by $X_{N,K}$.

Let H be a separable Hilbert space (Hilbert space with a countable basis) with corresponding norm $\|\cdot\|_H$. We usually take $H = L^2(\mathcal{D})$,

Definition 4.1 (Convergence)**.** *Assume that* $X_{N,K}$ *is a numerical solution and* $X(x, T)$ *is a solution at time* T*.*

- ***Mean-square convergence (Strong convergence)***. *If there exists a constant C independent of h and δ such that*
$$\mathbb{E}[\|X_{N,K} - X(\cdot,T)\|^2_{L^2(\mathcal{D})}] \leq C\big(h^{2p_1} + (\delta t)^{2p_2}\big), \quad p_1, p_2 > 0, \tag{4.1}$$
then the numerical solution is convergent in the mean-square sense to the solution. The mean-square convergence order in time is p_1 and the convergence order in physical space is p_2.
- ***Almost sure convergence (Pathwise convergence)***. *If there is a finite random variable $C(\omega) > 0$ independent of h and δ such that*
$$\|X_{N,K} - X(\cdot,T)\|_{L^2(\mathcal{D})} \leq C(\omega)\big(h^{p_1} + (\delta t)^{p_2}\big), \tag{4.2}$$
then the numerical solution is convergent almost surely to the solution.
- ***Weak convergence***. *If there exists a constant C independent of h and δ such that*
$$\|\mathbb{E}[\phi(X_{N,K})] - \mathbb{E}[\phi(X(\cdot,T))]\|_{L^2(\mathcal{D})} \leq C\big(h^{p_1} + (\delta t)^{p_2}\big), \tag{4.3}$$
then the numerical solution is weakly convergent to the solution.

We say the convergence order (in mean-square sense, almost sure sense or weak sense) in time is p_1 and the convergence order (in mean-square sense, almost sure sense or weak sense) in physical space is p_2.

Remark 4.2. *Here we do not specify what the sense of solutions is. The definition is universal for strong solutions, variational solutions and mild solutions for SPDEs.*

We do not consider the effect of truncation of infinite dimensional process W^Q. In general, the convergence of the truncated finite dimensional process to W^Q depends on the decay rate of q_i (2.3) as well as on the smoothing effect of the inverse of the leading operator.

References

[Lototsky and Rozovsky, 2017] Lototsky, S. V. and Rozovsky, B. L. (2017). *Stochastic partial differential equations*. Universitext. Springer, Cham.