LECTURE 7 STOCHASTIC ELLIPTIC EQUATIONS AND THEIR NUMERICAL METHODS

1. Elliptic equation with additive noise

$$-\Delta u + bu = \frac{\partial^d}{\partial x_1 \partial x_2 \cdots \partial x_d} W(x), \quad x = (x_1, \ldots, x_d) \in \mathcal{D} = (0,1)^d, \tag{1.1}$$

with $u = 0$ on $\partial\mathcal{D}$ and $b > 0$.

Here we represent the spatial white noise $\dot{W}(x)$ with an orthogonal series expansion

$$\frac{\partial^d}{\partial x_1 \partial x_2 \cdots \partial x_d} W(x) = \sum_{\alpha \in \mathcal{J}} e_\alpha(x) \xi_\alpha, \tag{1.2}$$

or for the spatial Brownian motion (Brownian sheet)

$$W(x) = \sum_{\alpha \in \mathbb{N}^d} \int_0^{x_d} \int_0^{x_{d-1}} \cdots \int_0^{x_1} e_\alpha(y)\, dy_1 \cdots dy_d \xi_\alpha, \tag{1.3}$$

where $\{e_\alpha(x)\}_{|\alpha|=1}^\infty$ is a complete orthonormal basis in $L^2(\mathcal{D})$; ξ_α, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_d)$ are independent standard Gaussian random variables. In practice, we can take any orthonormal basis in $L^2(\mathcal{D})$. Here we take the eigenfunctions of the elliptic equation

$$-\Delta\psi = \lambda\psi, \ x \in \mathcal{D} \quad \psi = 0, x \in \partial\mathcal{D}, \tag{1.4}$$

which can form an orthonormal basis in $L^2(\mathcal{D})$.

1.1. **Existence and Uniqueness.** Denote the eigenvalues of (1.4) by λ_α (which are proportional to $\alpha_1^2 + \cdots + \alpha_d^2$ and the corresponding eigenfunctions by e_α. Then by the method of the eigenfunction expansion,

$$u = \sum_{\alpha \in \mathbb{N}^d} \frac{\xi_\alpha}{b + \lambda_\alpha} e_\alpha. \tag{1.5}$$

Here $u \in \mathbb{L}^2(\Omega; L^2(\mathcal{D}))$ as

$$\mathbb{E}[\|u\|^2] = \mathbb{E}[\left\| \sum_{|\alpha|>0} \frac{\xi_\alpha}{b + \lambda_\alpha} e_\alpha \right\|^2] = \sum_{|\alpha|>0} \frac{1}{(b + |\alpha|^2)^2} \le C. \tag{1.6}$$

Recall the space

$$\dot{H}^s = \dot{H}^s(\mathcal{D}) = \mathcal{D}((-\Delta)^{s/2}) = \left\{ v \middle| \, \|v\|_s = \left\| (-\Delta)^{s/2} v \right\| = (\sum_{k=1}^\infty \lambda_k^s [(v, e_k)]^2)^{1/2} < \infty \right\}.$$

It is known that $\dot{H}^s = H^s(\mathcal{D})$, where $H^s(\mathcal{D})$ is the classical Sobolev-Hilbert space over $\mathcal{D}$.

We can show that $u \in \mathbb{L}^2(\Omega; H^{2-\frac{d}{2}-\epsilon}(\mathcal{D}))$.

(1.7)

$$\mathbb{E}[\|u\|_{\dot{H}^s}^2] = \mathbb{E}[\left\| \sum_{|\alpha|>0} \frac{\xi_\alpha (\lambda_\alpha)^{2s}}{b + \lambda_\alpha} e_\alpha \right\|^2] = \sum_{|\alpha|>0} \frac{(\lambda_\alpha)^{2s}}{(b + |\alpha|^2)^2} \le \sum_{|\alpha|>0} \frac{1}{(b + |\alpha|^2)^{2-2s}} \le C, \text{ where } s < 2 - \frac{d}{2}.$$

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1.2. **Approximation using a spectral representation.** We denote the truncation of $W(x)$ (1.3) by W_{n}:

$$W_{\mathsf{n}}(x) = \sum_{|\alpha|\leq \mathsf{n},\ \alpha\in\mathbb{N}^d} \int_0^{x_d}\int_0^{x_{d-1}}\cdots\int_0^{x_1} e_\alpha(y)\,dy_1\cdots dy_d\xi_\alpha. \tag{1.8}$$

A semi-discrete scheme is then

$$-\Delta u_{\mathsf{n}}(x) + bu_{\mathsf{n}} = \frac{\partial^d}{\partial x_1\partial x_2\cdots\partial x_d}W_{\mathsf{n}}(x) = \sum_{|\alpha|\leq \mathsf{n}} e_\alpha(x)\xi_\alpha. \tag{1.9}$$

Then we have $u_{\mathsf{n}} = \sum_{|\alpha|\leq\mathsf{n}} \frac{\xi_\alpha}{b+\lambda_\alpha}e_\alpha$. By (1.1) and (1.9), we have

$$\mathbb{E}[\|u-u_{\mathsf{n}}\|^2] = \mathbb{E}[\left\|\sum_{|\alpha|>\mathsf{n}+1}\frac{\xi_\alpha}{b+\lambda_\alpha}e_\alpha\right\|^2] = \sum_{|\alpha|>\mathsf{n}+1}\frac{1}{(b+|\alpha|^2)^2} \leq C\mathsf{n}^{-(4-d)/2}. \tag{1.10}$$

By the orthogonality of $e_\alpha(x)$ and independence of ξ_α's, we have the following weak convergence

$$\int_{\mathcal{D}} \mathbb{E}[u^2-u_{\mathsf{n}}^2]\,dx = \mathbb{E}[\|u\|^2 - \|u_{\mathsf{n}}\|^2] = \mathbb{E}[\|u-u_{\mathsf{n}}\|^2] \leq C\mathsf{n}^{-(4-d)/2}. \tag{1.11}$$

Remark 1.1. *We can also use a piecewise linear approximation (polygonal approximation) of the spatial Brownian motion (1D; For 2D and 3D, we can apply the same idea)*

$$W_h(x) = W(x_i) + (W(x_{i+1}) - W(x_i))\frac{x-x_i}{x_{i+1}-x_i}, \quad x\in[x_i,x_{i+1}). \tag{1.12}$$

The polygonal approximation will lead to different numerical methods. The mean-square convergence order is 1 *in 1D and* $2-d/2$ *in 2D and 3D if a central finite difference scheme is applied with a uniform step size.*

1.3. **Extensions.** The above approach can be further extended as follows:

- The domain $\mathcal{D}$ can be a bounded domain with a smooth boundary $\partial\mathcal{D}$, while conclusions remain true. For example, one can consider $\mathcal{D}$ is a bounded convex domain. See the treatment in [3, Chapter 4.2].
- The operator Δ can be replaced by general self-adjoint, positive-definite, linear operators, say $\mathcal{A}$, with compact inverse. For example,$-\mathcal{A}u = -\mathrm{div}(a(x)\nabla u)$, where $0 < a_m \leq a(x) \leq a_M$.
- The Gaussian white noise can be replaced by colored noise $\dot{W}^Q(x) = \sum_{k=1}\sqrt{q_k}m_k\xi_\alpha$. Also other processes can be used here as long as it has "smooth" solutions.

Nonlinear problems can also be considered.

$$-\Delta u(x) + f(u(x)) = g(x) + \frac{\partial^d}{\partial x_1\partial x_2\cdots\partial x_d}W(x), \quad x = (x_1,\cdots,x_d)\in\mathcal{D}, \tag{1.13}$$

with Dirichlet boundary condition

$$u(x) = 0, \quad x\in\partial\mathcal{D}, \tag{1.14}$$

where $\mathcal{D} = (0,1)^d$, $W(x)$ is a Brownian sheet on $\bar{\mathcal{D}} = [0,1]^d$, $g\in L^2(\mathcal{D})$ so that (1.1) is well-posed. The following assumption on the nonlinearity is a natural extension of Lipschitz continuity.

Assumption 1.2. *The function f satisfies the following conditions:*

- *There exists a constant $L < C_p$ such that*

$$[f(s)-f(t)](s-t) \geq -L\,|s-t|^2, \quad \forall s,t\in\mathbb{R}. \tag{1.15}$$

  Here C_p is the constant in the Poincare inequality

$$\|\nabla v\|^2 \geq C_p\,\|v\|^2, \quad v\in H_0^1(\mathcal{D}).$$

- *There exist constants $M \geq 0$ and $R \geq 0$ such that*

$$|f(s) - f(t)| \leq M + R\,|s-t|\,, \quad \forall s,t \in \mathbb{R}. \tag{1.16}$$

With Assumption 1.2, we have similar conclusions on regularity, strong and weak convergence.

Theorem 1.3 (Regularity)**.** *Under Assumption 1.2, we have the following regularity for the solution to* (1.1)*:*

$$\mathbb{E}[\|u\|_{L^p}^q] < \infty, \quad p > 1,\, q \geq 1. \tag{1.17}$$

Furthermore, if $M = 0$ in (1.16) *of Assumption 1.2,*

$$\mathbb{E}[\|u\|_{2-d/2-\epsilon}^2] < C\epsilon^{-1}. \tag{1.18}$$

Theorem 1.4 (Mean-square convergence)**.** *Let u be the solution to* (1.1) *and u_{n} the solution to* (1.9)*. Under Assumption 1.2,*

$$\mathbb{E}[\|u - u_{\mathsf{n}}\|^2] \leq C\big[M\mathsf{n}^{-(2-d/2)} + (C_p + R)^2\mathsf{n}^{-(4-d)}\big]. \tag{1.19}$$

Theorem 1.5. *(Weak convergence) Assume that f and F and their derivatives up to fourth-order are of at most polynomial growth at infinity, i.e., they are bounded by*

$$c(1 + |x|^\kappa), \quad 1 \leq \kappa < \infty.$$

The rate of the weak convergence is twice the strong convergence order:

$$\|\mathbb{E}[F(u)] - \mathbb{E}[F(u_{\mathsf{n}})]\|_{L^q} \leq C \sum_{|\alpha|=\mathsf{n}+1}^{\infty} \lambda_\alpha^{-2} \leq C\mathsf{n}^{-(4-d)}, \quad 1 \leq q < \infty. \tag{1.20}$$

For details of treatment, readers are referred to [4, Chapter 10].

2. Abstract framework for stability of solutions to elliptic equations

A Banach space Y is continuously embedded into a Banach space X if Y is a subset of X and there exists a number $c > 0$ such that $\|y\|_X \leq c\,\|y\|_Y$ for all $y \in Y$. It can be shown that if a Banach space Y is a sub-set of a Banach space X and define the embedding operator $I : Y \to X$ by $Iy = y$. Show that Y is continuously embedded into X if and only if the operator I is continuous.

Definition 2.1 (Normal triple)**.** *A normal triple of Hilbert spaces is an ordered collection of (V, H, V') of three Hilbert spaces with the following properties:*

- *1. V is a dense sub-set of H and is continuously embedded into H,*
- *2. H is dense sub-set of V' and is continuously embedded into H,*
- *3. For all $v \in V, h \in H$ it holds that $|(v,h)_H| \leq \|v\|_V\|h\|_{V'}$*

In this course, we will mostly use (H_0^1, L^2, H^{-1}).

Theorem 2.2 (Solvability of elliptic equations and stability)**.** *Let (V, H, V') be a normal triple of Hilbert spaces and let $\mathrm{A} : V \to V'$ be a bounded linear operator with the following property: there exists a positive number c_A such that, for all $v \in V$, $[\mathrm{A}v, v] \geq c_A\|v\|_V^2$. Then, for every $f \in V'$, there exists a unique $u \in V$ such that $Au = f$ holds in V'. Moreover, $\|u\|_V \leq \frac{1}{c_A}\|f\|_{V'}$*

The proof is via the Lax-Milgram theorem by defining the bilinear form $a(u,v) = [Au, v] = [f, v]$.

3. Elliptic equation with multiplicative noise

Let's consider the following elliptic problem with multiplicative noise.

$$-\,\mathrm{div}(e^W \nabla u) + r(x)u = f, \quad x \in \mathcal{D} \subseteq \mathbb{R}^d. \tag{3.1}$$

where $W(x)$ is the Brownian sheet on $\mathcal{D}$ and a vanishing Dirichlet boundary condition is imposed. Here $r(x) \geq 0$ and f are continuous on $\mathcal{D}$ and both are non-random.

3.1. **Existence and uniqueness.** Let's assume for simplicity $r \equiv 0$. Denote $a_m = \min_{x\in D} e^{W(x)}$ and $a_M = \max_{x\in D} e^{W(x)}$. By the stability of the elliptic equations, we have

$$\|u\|_{H_0^1} \leq a_m^{-1} \|f\|_{H^{-1}}. \tag{3.2}$$

By the fact that $\mathbb{E}[a_m^{-p}] < \infty$, $1 \leq p < \infty$ (See [1] for a proof), we have

$$\mathbb{E}[\|u\|_{H_0^1}^p] \leq \mathbb{E}[a_m^{-p}] \|f\|_{H^{-1}}^p, \quad 1 \leq p < \infty. \tag{3.3}$$

If f is random, we have

$$\mathbb{E}[\|u\|_{H_0^1}^p] \leq (\mathbb{E}[a_m^{-2p}])^{1/2}(\mathbb{E}[\|f\|_{H^{-1}}^{2p}])^{1/2}, \quad 1 \leq p < \infty. \tag{3.4}$$

An explicit solution to (3.1) can be hardly found except in very special cases. For example, when $\mathcal{D} = (0,1)$ $(d = 1)$, the solution takes the form

$$u(x) = \int_0^x e^{-W(y)} \int_0^y f(z)\, dz\, dy - x \int_0^1 e^{-W(y)} \int_0^y f(z)\, dz\, dy.$$

3.2. **An semi-discrete approximation using a spectral approximation.** In computation, we work with the following semi-discrete system

$$-\operatorname{div}(a_{\mathsf{n}}(x)\nabla u_{\mathsf{n}}) = f(x), \quad x \in \mathcal{D}, \qquad u_{\mathsf{n}}(x) = 0, \quad x \in \partial\mathcal{D}, \tag{3.5}$$

where a_{n} is defined in (1.8). A natural question of the convergence of this model to (3.10).

Let us derive the convergence of the semi-discrete model. The difference of the two equations reads (sometimes it is called the error equation)

$$-\operatorname{div}(a_{\mathsf{n}}(x)\nabla(u - u_{\mathsf{n}})) = \operatorname{div}((a(x) - a_{\mathsf{n}}(x))\nabla u), \quad x \in \mathcal{D}, \qquad u_{\mathsf{n}}(x) = 0, \quad x \in \partial\mathcal{D}.$$

Then by the stability of the elliptic equation, we have

$$\begin{aligned}
\mathbb{E}[\|u - u_{\mathsf{n}}\|_{H_0^1(\mathcal{D}))}^p] &\leq C(\mathbb{E}[(\min_{x\in\mathcal{D}} a_{\mathsf{n}})^{-2p}])^{1/2}(\mathbb{E}[\|(a_{\mathsf{n}} - a)\nabla u\|_{H^1(\mathcal{D})))}^{2p}])^{1/2} \\
&\leq C(\mathbb{E}[\|(a_{\mathsf{n}} - a)\nabla u\|_{L^2(\mathcal{D})))}^{2p}])^{1/2} \\
&\leq C\mathbb{E}[\|a_{\mathsf{n}} - a\|_{L^\infty(\mathcal{D})}^{2p}]\mathbb{E}[\|u\|_{H^1(\mathcal{D})))}^{2p}] \leq C\mathbb{E}[\|a_{\mathsf{n}} - a\|_{L^\infty(\mathcal{D})}^{2p}].
\end{aligned}$$

Remark 3.1. *Here we can also apply the stability theory in Hölder spaces and have (see* [1]*)*

$$\mathbb{E}[\|u - u_{\mathsf{n}}\|_{\mathcal{C}^{1,\nu}(\bar{\mathcal{D}}))}^p] \leq C(\mathbb{E}[\|a_{\mathsf{n}} - a\|_{\mathcal{C}^{0,\nu}(\bar{\mathcal{D}})}^{2p}])^{1/2} \qquad q \geq 1. \tag{3.6}$$

Now we can apply finite difference schemes to further discretize (3.5) (with the term $r(x)u$). For example, we can apply the following central difference scheme.

$$-\frac{a_{\mathsf{n},j+\frac{1}{2}}(u_{j+1} - u_j) - a_{\mathsf{n},j-\frac{1}{2}}(u_j - u_{j-1})}{h^2} + r_j u_j = f_j, \quad j = 1, 2, \ldots, n-1.$$

where $a_{\mathsf{n},j\pm\frac{1}{2}} = e^{W_{\mathsf{n}}(x_{j\pm\frac{1}{2}})}$. The resulting linear system is

$$\mathbf{A}\vec{u} = \vec{f} + \vec{g},$$

where $\vec{u} = (u_1, u_2, \ldots, u_{n-1})^\top$, $\vec{f} = (f(x_1), f(x_2), \ldots, f(x_{n-2}), f(x_{n-1}))^\top$, $\vec{g} = (0, 0, \ldots, 0, 0)^\top$ and

$$\mathbf{A} = \begin{pmatrix}
\bar{a}_{\mathsf{n},1} + h^2 r(x_1) & -a_{\mathsf{n},\frac{3}{2}} & & \\
-a_{\mathsf{n}\frac{3}{2}} & \bar{a}_{\mathsf{n},2} + h^2 r(x_2) & -a_{\mathsf{n}\frac{5}{2}} & \\
\ddots & \ddots & \ddots & \\
& -a_{\mathsf{n},\frac{5}{2}} & \bar{a}_{\mathsf{n},n-2} + h^2 r(x_{n-2}) & -a_{\mathsf{n},\frac{2n-3}{2}} \\
& & -a_{\mathsf{n},\frac{2n-3}{2}} & \bar{a}_{\mathsf{n},n-1} + h^2 r(x_{n-1})
\end{pmatrix}.$$

Here $\bar{a}_{\mathsf{n},j} = a_{\mathsf{n},j+\frac{1}{2}} + a_{\mathsf{n},j-\frac{1}{2}}$ Here we eliminate u_n and $\mathbf{A}$ is a tridiagonal matrix and is diagonally dominant if $r(x) > 0$.

Remark 3.2. *When $a \equiv 1$ and $r \equiv 0$, the eigenvalues of* $\mathbf{A}$ *are* $h^{-2}(2+2\cos(k\pi/n)))$, $k = 1, 2, \cdots, n-1$.

Remark 3.3. *We may apply the finite difference scheme for* (3.1) *instead of for* (3.5).

Here $\mathbf{A}(\omega)$ is a random matrix. We need to solve for each ω a linear system. Is it possible to avoid the inverse of randomize matrix? Let's try another formulation of (3.1). By multiplying e^{-W} over both sides of the equation, we have

$$-\Delta u + \nabla W \cdot \nabla u + r(x)e^{-W}u = e^{-W}f, \quad x \in \mathcal{D} \subseteq \mathbb{R}^d. \tag{3.7}$$

Remark 3.4 (Mild solution)**.** *Suppose that* $K(x, y)$ *is the Green function on* $\mathcal{D}$ *for* $-\Delta$. *Then we have the following mild solution*

$$u + \int_{\mathcal{D}} K(x,y)r(x)e^{-W}u\,dx = -\int_{\mathcal{D}} K(x,y)\nabla W \cdot \nabla u\,dx + \int_{\mathcal{D}} K(x,y)e^{-W}f\,dx, \quad x \in \mathcal{D} \subseteq \mathbb{R}^d. \tag{3.8}$$

Here we need to interpret the product between ∇W and ∇u. Should it be Ito or Stratonovich or something else? The intuition here is that we have applied the chain rule as usually done in deterministic calculus and thus we should have the Stratonovich product here.

We may build a finite difference scheme based on (3.7). In one dimension, we may have

$$-\frac{(u_{j+1}-u_j)-(u_j-u_{j-1})}{h^2} + \frac{W(x_{j+1})-W(x_j)}{h}\frac{u_{j+1}-u_j}{h} = e^{-W(x_j)}f_j, \quad j = 1, 2, \dots, n-1.$$

or

$$-\frac{(u_{j+1}-u_j)-(u_j-u_{j-1})}{h^2} + \frac{W(x_{j+1})-W(x_j)}{h}\frac{u_{j+1}-u_{j-1}}{2h} = e^{-W(x_j)}f_j, \quad j = 1, 2, \dots, n-1.$$

Which one is right?

One way to obtain a correct answer is through the semi-discrete approximation (3.5). Note that $W_{\mathsf{n}}(x)$ is a smooth process approximating $W(x)$ and an equivalent formulation of (3.5) is

$$-\Delta u_{\mathsf{n}} + \nabla W_{\mathsf{n}} \cdot \nabla u_{\mathsf{n}} + r(x)e^{-W_{\mathsf{n}}}u_{\mathsf{n}} = e^{-W}f, \quad x \in \mathcal{D} \subseteq \mathbb{R}^d. \tag{3.9}$$

This is a Wong-Zakai approximation of (3.10). Thus, the product between $\nabla W(x)$ and ∇u should be interpreted as the Stratonovich product and the discretization should involve some symmetry.

3.3. **Extensions.** The problem (3.1) can be extended in many ways. Let's consider the following form.

$$-\operatorname{div}(a(x,\omega)\nabla u(x,\omega)) = f(x), \quad x \in \mathcal{D}, \qquad u(x) = 0, \quad x \in \partial\mathcal{D}, \tag{3.10}$$

where $\mathcal{D}$ is a domain with Lipschitz boundary, $a(x, \omega)$ is a random field, and f is either random or deterministic.

Following extensions on $a(x)$ are possible.

- Assume that $\ln(a(x, \omega))$ is a Gaussian field with zero mean and covariance kernel $K(x-y)$, e.g., $K(x-y) = \sigma^2 \exp(\frac{|x-y|^p}{l^2})$, $p = 1, 2$ or any real number that $K(x-y)$ is square-integrable. (later on we need a finite dimensional approximation of $a(x, \omega)$.)

  Clearly, we can apply the e Karhunen-Loève expansion if K is positive-definite. For example, we can use the following representation

  $$a(x,\omega) = \exp\big(\sigma \sum_{k=1}^{\infty} \lambda_k^{1/2}\phi_k(x)\xi_k(\omega)\big), \tag{3.11}$$

  where λ_k are nonnegative real numbers, $\phi_k(x)$ is a CONS in $L^2(\mathcal{D})$,and ξ_k's are i.i.d. standard Gaussian random variables.
- If $a(x, \omega)$ is a bounded stochastic field, i.e., for a.e. ω, $a(x, \omega)$ is uniformly bounded in x ($a(x, \omega$ is uniformly bounded in both x and ω.), then no significant issues will arise (can be treated "deterministically"). For example, $a(x) = a_0(x) + \sum_{k=1}^{\infty} a_k(x)\zeta_k$, where $a_0(x) > a_0$, a_k's are uniformly bounded and $\zeta_k \sim \mathcal{U}(0, 1)$.

Let's consider the problem when $a(x)$ is lognormal.

Theorem 3.5 (Existence and uniqueness)**.** *Assume that $\mathcal{D}$ is an open bounded domain in $\mathbb{R}^d$ with $\mathcal{C}^2$ boundary. When $f \in L^2(\mathcal{D})$ and $K(\cdot)$ is in $\mathcal{C}^{0,1}(\mathbb{R}^+)$, there exists a unique solution (a.e.) in $\mathbb{L}^q(\Omega, H_0^1(\mathcal{D}))$.*

When $K(z)$ is in $\mathcal{C}^{0,1}(\mathbb{R}^+)$, a and $\ln(a)$ belong to $\mathcal{C}^{0,\mu}(\bar{\mathcal{D}})$ a.s. for $\mu < 1/2$. (See Problem 8, Homework 2.) Below are two examples.

- $e^{W(x)}$ with $\lambda_k \sim 1/k^2$ in 1D. $\mathbb{E}[W(x)W(y)] = \min(x, y)$.

$$e^{W(x)} = \exp\big(\sum_{k=1}^{\infty} \int_0^x m_k(y)\, dy \xi_k(\omega)\big)$$

- $\ln(a(x))$ as a Gaussian process in one-dimension with exponential kernel $K(x-y) = \exp(-\frac{|x-y|}{l_c})$.

Now we truncate a with a_{n} as

$$a_{\mathsf{n}} = e^{\sigma^2/2} \exp(\sigma \sum_{k=1}^{\mathsf{n}} \lambda_k^{1/2} \phi_k(x) \xi_k)$$

For any μ, ν with $0 \le \nu < \mu < 1/2$ and $q \ge 1$,

$$\|a_{\mathsf{n}} - a\|_{\mathbb{L}^q(\Omega, \mathcal{C}^{0,\nu}(\bar{\mathcal{D}}))} \le C(R_{\mathsf{n}}^{\mu})^{1/2}, \quad R_{\mathsf{n}}^{\mu} = \sum_{k=\mathsf{n}+1}^{\infty} \lambda_k \|\phi_k\|_{\mathcal{C}^{0,\mu}}^2 .$$

where the positive C depends only on μ, ν, q.

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Theorem 3.6. *For $f \in L^p(\mathcal{D})$, $p \ge d$, $0 < \nu < \min(\frac{1}{2}, 1 - \frac{d}{p})$, the strong convergence order is*

$$\mathbb{E}[\|u - u_{\mathsf{n}}\|_{\mathcal{C}^{1,\nu}(\bar{\mathcal{D}}))}^q] \le C(R_{\mathsf{n}}^{\nu})^{\frac{q}{2}}, \qquad q \ge 1. \tag{3.12}$$

(Weak convergence) If $\psi \in \mathcal{C}^6(\mathbb{R})$ and ψ and its derivatives have at most polynomial growth at ∞,

$$\|\mathbb{E}[\psi(u)] - \mathbb{E}[\psi(u_{\mathsf{n}})]\|_{\mathcal{C}^{1,\nu}(\bar{\mathcal{D}})} \le C R_{\mathsf{n}}^{\nu},$$

where the positive constant C depends only on β, p and f, ψ.

Example 3.7. *For Brownian motion (spatial) over $(0, L)$,*

$$W(x) = \sum_{k=1}^{\infty} \int_0^x e_k(y)\, dy \xi_k, \qquad e_1(x) = \sqrt{\frac{1}{L}},\, e_k(x) = \sqrt{\frac{2}{L}} \cos(\frac{k\pi x}{\sqrt{L}}),\, k \ge 2,$$

$\lambda_k \sim \mathcal{O}(\frac{1}{k^2})$ and $\|\phi_k\|_{\mathcal{C}^{0,\nu}([0,L])} \sim \mathcal{O}(k^{\nu})$ and thus $R_{\mathsf{n}}^{\nu} \sim \mathcal{O}(N^{2\nu-1})$.

For further discretize the semi-discrete approximation, finite difference methods in physical space and Monte Carlo methods can be applied.

Alternative methods in physical space and/or in random space can be

- Finite element methods, many subclass methods of finite element methods, spectral methods, finite volume method, meshfree methods, spline methods, etc.
- Quasi-Monte Carlo methods, stochastic collocation methods, stochastic Galerkin methods etc, multilevel version of these methods,
- Model reduction methods can be also applied in both physical and random space.

References

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