Lec. 6
Introduction to SPDEs

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- PDEs + random ingredients
  - random initial, boundary conditions or random coefficients
  - random domain (e.g., boundary is random), including random terminal time
  - random driving force

Recall deterministic PDEs: for some functional relation possibly including all independent variables.

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Stochastic PDEs

$$\underbrace{F(u, Du, D^2u, \cdots)}_{\text{deterministic part}} = \underbrace{G(u, Du, D^2u, \cdots) \cdot \text{Noise}}_{\text{stochastic part}}.$$

Here F and G are random functions.

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  - Gaussian white noise
  - Gaussian fractional noise
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Classification of SPDEs: I. as SDEs

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  - Levy processes. e.g. Poisson processes, α-stable processes
- the manner the noise entering the equation
  - additive noise (G does not depend on u and any derivatives)
  - multiplicative noise (otherwise)
- the type of stochastic integral
  - Ito integral (and extensions)
  - Stratonovich integral
  - $\ldots$

- the order of the equation;
- the type of the nonlinearity in the equation;
- the type of the initial and boundary conditions;
- elliptic/hyperbolic/parabolic.

• *The order of an SPDE* is the highest order of the partial derivative "appearing" in the equation. This order can depend on the type of the stochastic integral.

Example

Consider the following first-order Stratonovich equation

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$$du(t,x) = u_x(t,x) \circ dw(t) \tag{1}$$

$$du(t,x) = \frac{1}{2} u_{xx}\, dt + u_x(t,x) dw(t) \text{ (2nd-order!)} \tag{2}$$

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- (quasilinear) both F and G are linear in highest-order derivatives but coefficients depending on u and lower-order derivatives.

$$du = (u_x u_{xx} - u^3)\, dt + u u_x \dot{W}(t), \quad x \in (-1, 1).$$

- (fully nonlinear) both F and G are nonlinear in highest-order derivatives

$$du = (u_x)^2 \circ \dot{W}(t), \quad x \in (-1, 1).$$

Remark on bilinear equations

$$du(t,x) = au_{xx}\,dt + \sigma u_x(t,x)dw(t). \tag{3}$$

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• Bilinear equations are much more difficult to study than linear equations with additive noise:

- Solutions are not Gaussian processes.
- Fundamental solutions are random (be careful with variation-of-constant-coefficients)
- Interactions between the stochastic part and the deterministic part. For example the equation is well-posed in $\mathbb{L}^2(\Omega; L^2(\mathbb{R}))$ if and only if $2a > \sigma^2$.

• (**forward problems**) well-posedness of the equation: existence, uniqueness, and continuous dependence on the input data.

- Markov properties of the solution
- properties of the solution as a random variable (existence and regularity of the density and various statistical moments),
- asymptotic problems (small parameter: large deviations, averaging, and homogenization;large time behavior: existence of attractors and invariant measures, ergodicity, stability)
- fine properties of the sample paths (local times, level sets, potential theory, small ball probabilities)
- methods of computing the solution numerically.

• **(inverse problems)** : determining (some of) the input data from the observations of the solution of the equation
Inverse problems are usually solved using the methods of statistical inference.

• **(synthesis problems)**: optimal filtering and optimal control of processes governed by SPDEs

Notions of solutions

A solution of an SPDE can be

• either strong (constructed on a given probability space)

• or weak (the probability space is constructed as part of the solution). Analogue to SODEs.

We will focus on strong solutions.

In the PDE sense the solution of an SPDE can be

- classical;
- generalized:
  - closed-form;
  - mild;
  - variational;
    - strong;
    - weak;
    - measure-valued;
    - chaos;
  - viscosity; (Not covered)

• A classical solution of an SPDE is a continuous function satisfying the equation and the initial and boundary conditions point-wise on the same set of probability one.
if the noise is not regular in space, the equation is unlikely to have a classical solution.

• Every solution that is not classical is usually called generalized.

◇ A mild solution is an extension of the closed-form solution and is mostly used for equations that can be considered perturbations of a linear equation with a closed-form solution

$$v_t(t,x) = v_{xx}(t,x) + f(t,x), t > 0, x \in \mathbb{R}, \tag{4}$$

The closed-form solution of this equation is

$$v(t,x) = \int_{\mathbb{R}} \Phi(x-y,t)v(0,y)dy + \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)f(s,y)dyds$$

$$du(t,x) = \left(u_{xx} - u^3(t,x)\right) dt + \sin(u(t,x))dw(t) \tag{5}$$

The mild-solution is then

$$\begin{aligned} v(t,x) &= \int_{\mathbb{R}} \Phi(x-y,t)v(0,y)dy + \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)u^3(s,y)dyds \\ &+ \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)\sin(u(s,y))dydW(s). \end{aligned}$$

$$v(t,x) = e^{\partial_x^2 t} v(0,y) + \int_0^t e^{\partial^2(t-s)} u^3(s)\, ds + \int_0^t e^{\partial^2(t-s)} \sin(u(s))\, dW(s).$$

Consider the equation $\partial_t u(t,x) = u_{xx}(t,x)$ and $u(0,x) = u_0(x)$.

•**Strong variational solution** is a function u from $L^1([0,T] : H^2(\mathbb{R}))$ such that

$$u(t,x) = f(x) + \int_0^t u_{xx}(s,x)ds \tag{6}$$

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• **Weak variational solution I** is a function u from $L^1([0,T]:H^1(\mathbb{R}))$ such that, for every smooth function $v = v(x)$ with a compact support in $\mathbb{R}$, the equality

$$(u(t,\cdot),v)_{L_2(\mathbb{R})} = (f,v)_{L_2(\mathbb{R})} - \int_0^t (u_x(s,\cdot),v_x)_{L_2(\mathbb{R})}\,ds \tag{7}$$

- **Weak variational solution II** is a function u from $L^1([0,T];L^2(\mathbb{R}))$ such that, for every smooth function $v = v(x)$ with a compact support in $\mathbb{R}$, it holds for all $t \in (0,T]$

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- **Weak variational solution III** is a function u locally integrable on $[0,T]\times\mathbb{R}$ such that, for every smooth function $v = v(x)$ with a compact support in $[0,T]\times\mathbb{R}$, it holds

$$\int_{\mathbb{R}} v(0,x)f(x)dx + \int_0^T \int_{\mathbb{R}} u(t,x)\left(v_t(t,x) + v_{xx}(t,x)\right) dxdt = 0 \tag{9}$$

• **(Measure-valued solution)** is a collection of sigma-finite signed measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that, for every smooth function $\phi(x)$ compactly supported on $\mathbb{R}$,

$$\mu_t[\varphi] = \int_{\mathbb{R}} \varphi(x) f(x) dx + \int_0^t \mu_s [\varphi_{xx}] \, ds, \quad \mu_t[g] = \int_{\mathbb{R}} g(x) \mu_t(dx). \quad (10)$$

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• **(Chaos solution)** (stochastic Galerkin)
Suppose that $h(t) \in C_0^\infty([0,T])$. The test functions are the exponential martingale

$$M_h = \exp\Big(\int_0^T h(t)\, dW(t) - \frac{1}{2}\int_0^T h^2(t)\, dt\Big). \quad (11)$$

Taking h from a CONS form a dense basis in the random space.

$$u_h = \mathbb{E}[uM_h], \quad u = \sum_{k=1}^{\infty} u_{h_k} M_{h_k}.$$

Here u is called the Chaos solution

$$du(t,x) = \frac{1}{2} u_{xx}(t,x)\, dt + u_x\, dW(t), \quad 0 < t < T. \quad (12)$$

$$du_h(t,x) = \frac{1}{2} u_{h,xx}(t,x)\, dt + u_{h,x} h(t), \quad 0 < t < T. \quad (13)$$

$$dv(t,x) = \frac{1}{2} v_{xx}(t,x)\, dt \tag{14}$$

Let $u(t,x) = v(t, x + w(t))$. Then

$$\begin{aligned} du(t,x) &= v_t(t, x + w(t))dt + v_x(t, x + w(t))dw(t) + \frac{1}{2} v_{xx}(t, x + w(t))dt \\ &= u_{xx}(t,x)\, dt + u_x(t,x)\, dw(t) \end{aligned} \tag{15}$$

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Let $u(t,x) = v(t, x + w(t))$. Then

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Let's consider

$$du(t,x) = u_{xx}(t,x)\, dt + u_x(t,x)\, dw(t)$$

Define $v(t,x) = u(t, x - w(t))$. Applying Ito's formula, we have

$$dv = u_t dt - u_x dw + \frac{1}{2} u_{xx}\, dt, \text{ or } v_t = \frac{3}{2} v_{xx}.$$

- u is not differentiable in t. New stochastic calculus needed.

$$dF(t,x) = J(t,x)\,dt + H(t,x)\,dw(t)$$
$$dY(t) = b(t)\,dt + \sigma(t)\,dw(t)$$

Theorem (Ito-Wentzell formula I)

$$\begin{aligned} dF(t, Y(t)) &= J(t, Y(t))dt + H(t, Y(t))dw(t) \\ &\quad + F_x(t, Y(t))\,dY(t) + \frac{1}{2}\sigma^2(t)F_{xx}(t, Y(t))dt \\ &\quad + H_x(t, Y(t))\sigma(t)dt \end{aligned}$$

$$\begin{aligned} dF(t, Y(t)) &= \left. dF(t,x)\right|_{x=Y(t)} \\ &\quad + F_x(t, Y(t))dY(t) + \frac{1}{2}F_{xx}(t, Y(t))d\langle Y\rangle(t) \\ &\quad + d\left\langle F_x, Y\right\rangle(t) \end{aligned}$$

$$dF(t, Y(t)) = J(t, Y(t))dt + H(t, Y(t)) \circ dw(t) + F_x(t, Y(t)) \circ dY(t)$$

$$dF(t, Y(t)) = dF(t, x) \mid_{x=Y(t)}$$

$$+ \sum_{i=1}^{d} D_i F(t, Y(t)) dY_i(t) + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{m} D_i D_j F(t, Y(t)) \sigma_{ik}(t) \sigma_{jk}(t) dt$$

$$+ \sum_{i=1}^{d} \sum_{k=1}^{m} D_i H_k(t, Y(t)) \sigma_{ik}(t) dt$$

Go back to the previous example.

$$du(t,x) = u_{xx}dt + u_x dw(t),$$

$$du_x = u_{xxx}dt + u_{xx}dw$$

$$d\left\langle u_x, w\right\rangle_t = u_{xx}dt,$$

Applying Ito-Wentzell formula, $v(t,x) = u(t, x - w(t))$ becomes

$$dv = du - u_x dw(t) + \frac{1}{2}u_{xx}dt - u_{xx}dt = \frac{1}{2}v_{xx}dt,$$

- bilinear equations by Fourier transform and integrating-factor method
- heat equation with additive space-time white noise
- elliptic equation

See more at Chapter 2.3 of Lototsky and Rozovsky , 2017

- Variation of constant-coefficients, integrating factor methods
- Stochastic transformation methods

Consider the following stochastic Burgers equation on $(0, T] \times (0, 1)$:

$$\partial_t u + u\partial_x u = \mu\partial_x^2 u + \sigma(t, x)\dot{W}(t), \quad u(0, x) = u_0(x),\ u(t, 0) = u(t, 1). \tag{16}$$

If $\sigma(t, x)$ depends only on t, then the solution is

$$u(t, x) = v(t, x - \int_0^t \sigma(s)W(s)\, ds) + \int_0^t \sigma(s)W(s)\, ds, \tag{17}$$

where $v(t, x)$ satisfies the deterministic Burgers equation

$$\partial_t v + vv_x = \mu\partial_x^2 v, \quad v(0, x) = u_0(x),\ v(t, 0) = v(t, 1). \tag{18}$$

The stochastic Korteweg-de Vries (KdV) equation

$$\partial_t u + u\partial_x u + \mu\partial_x^3 u + \gamma u = f(t,\omega) \tag{19}$$

can be transformed to the following KdV equation on $(0, T]$

$$\partial_t v + v\partial_x v + \mu\partial_x^3 v + \gamma v = 0, \tag{20}$$

and $u(t,x) = v(t, x - \int_0^t g(s,\omega)\, ds) + g(s,\omega)$, where

$$g(s,\omega) = e^{-\gamma t} \int_0^t e^{\gamma s} f(s,\omega)\, ds.$$

The Navier-Stokes equation with additive random forcing also can be transformed into a deterministic one. Through the substitutions $u(t,x) = U(t, x - \int_0^t g(s,\omega)\,ds)$ and $p = P(t, x - \int_0^t g(s,\omega)\,ds)$, $g(s,\omega) = \int_0^t f(s,\omega)\,ds$, we obtain from

$$\partial_t u + u\nabla u = \mu\Delta u - \nabla p + f(t,\omega). \tag{21}$$

that

$$\partial_t U + U\nabla U = \mu\Delta U - \nabla P. \tag{22}$$

Though the boundary conditions for (21) may be different from those for (22), the initial conditions are the same.

• If we are given periodic boundary conditions for (21), we can still apply periodic boundary conditions to (22).

For multiplicative noise, we can apply similar techniques if the noise is only time-dependent.
Consider the advection-diffusion equation

$$\partial_t u + f(t,\omega)\partial_x u = \mu \partial_x^2 u. \quad \mu \geq 0.$$

Let $u(t,x,\omega) = v(t, x - \int_0^t f(s,\omega)\,ds)$. Then we have

$$\partial_t v = \mu \partial_x^2 v.$$

• If $f(t,\omega)$ is white noise, we have to interpret the product $f(t,\omega)\partial_x u$ using Stratonovich product: $f(t,\omega) \circ \partial_x u$ as in (23).

• Try also the integrating factor method.

The solution to the following Stratonovich Burgers equation on $(0, T] \times (0, 1)$

$$\partial_t u + (u + \sigma \dot{W}(t)) \circ \partial_x u = \mu \partial_x^2 u, \quad u(0, x) = u_0(x),\ u(t, 0) = u(t, 1) \tag{23}$$

is given by

$$u(t, x) = v(t, x - \sigma W(t)), \tag{24}$$

where $v(t, x)$ satisfies Equation (18).
Since we usually don't have analytical solutions to Equation (18), we first find a numerical solution for v in Equation (18) and then obtain the solution.

More in details in the rest of classes.
See a review at Chapter 3.4 of Zhang and Karniadakis, 2017