LECTURE 5 CONNECTIONS TO PDES AND SPDES

1. Linear mean-square stability of some schemes

For fixed h, let's analyze the behavior of numerical schemes when $n \to \infty$. This is done by analyzing the simple but a revealing linear test problem

$$\begin{aligned} \mathrm{d}X(t) &= \lambda X(t)\mathrm{d}t + \sigma X(t)\, dW(t), \\ X(t) &= x, \end{aligned} \tag{1.1}$$

where $\lambda < 0$, $\sigma \geq 0$. It can be shown that when $2\lambda + \sigma^2 < 0$, the solution to (1.1) is asymptotically stable in the mean-square sense:

$$\lim_{t\to\infty} \mathbb{E}[X^2(t)] = 0, \quad \text{where } X(t) = \exp((\lambda - \frac{1}{2}\sigma^2)t + \sigma W(t)).$$

1.1. **Stability region of the forward Euler scheme.** Applying the forward Euler scheme to the linear test equation (1.1), we have

$$X_{n+1} = (1+z)X_n + \sqrt{y}X_n\xi_n, \quad z = \lambda h, \text{ and } y = \sigma\sqrt{h}.$$

Then $\mathbb{E}[X_{n+1}^2] = \mathbb{E}[(1+z+y\xi_n)^2]\mathbb{E}[X_n^2]$, and thus

$$\mathbb{E}[X_{n+1}^2] = ((1+z)^2 + y^2)\mathbb{E}[X_n^2].$$

By the discrete Gronwall inequality, we need $(1+z)^2 + y^2 < 1$ so that $\mathbb{E}[X_{n+1}^2]$ is decreasing and is asymptotically mean-square stable. Thus, the stability region is

$$\left\{(z,y) \in (-2,0)\times(0,1) | (1+z)^2 + y^2 < 1\right\}.$$

In the following, it is shown that the stability region of the midpoint scheme is lager than the forward Euler scheme.

1.2. **Stability analysis of the midpoint scheme.** In Stratonovich form, (1.1) is written as

$$\mathrm{d}X(t) = \tilde{\lambda}X(t)\mathrm{d}t + \sigma X(t)\circ \mathrm{d}W_l(t), \quad X(0) = x. \tag{1.2}$$

where $\tilde{\lambda} = \lambda - \frac{\sigma^2}{2}$. The midpoint scheme applied to (1.2) reads

$$X_{n+1} = X_n + \tilde{\lambda}h(X_n + X_{n+1})/2 + \sigma(X_{n+1} + X_n)/2\zeta_{h,n}\sqrt{h}, \quad n \geq 0, \tag{1.3}$$

where $\zeta_{h,n}$ are i.i.d. random variables so that

$$\zeta_h = \begin{cases} \xi, \ |\xi| \leq A_h, \\ A_h, \ \xi > A_h, \\ -A_h, \ \xi < -A_h, \end{cases} \tag{1.4}$$

From (1.3), we have $X_{n+1} = R(\lambda h, \sigma\sqrt{h})X_n$ where

$$R(\tilde{\lambda}h, \sigma\sqrt{h}) = \frac{1 + \tilde{\lambda}h/2 + \sigma\zeta_{h,n}\sqrt{h}}{1 - \tilde{\lambda}h/2 - \sigma\zeta_{h,n}\sqrt{h}}.$$

To have an asymptotically stable solution in the mean-square sense, we need $\mathbb{E}[R^2(\tilde{\lambda}h, \sigma\sqrt{h})] < 1$ such that

$$\mathbb{E}[X_{n+1}^2] = \mathbb{E}[R^2(\tilde{\lambda}h, \sigma\sqrt{h})]\mathbb{E}[X_n^2] < \mathbb{E}[X_n^2].$$

This requires that

$$\mathbb{E}[R^2(\tilde{\lambda}h, \sigma\sqrt{h})] < 1.$$

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Table 1. Mean-square stability regions of three different schemes

scheme	additive noise	multiplicative noise
Euler	$\{(z,y) \in (-2,0) \times (0,\infty)\}$	$\{(z,y) \in (-2,0) \times (0,1) \vert (1+z)^2 + y^2 < 1\}$
backward Euler	$\{(z,y) \in (-\infty,0) \times (0,\infty)\}$	$\{(z,y) \in (-\infty,0) \times (0,\infty) \vert 1 + y^2 < (1-z)^2\}$
midpoint scheme	$\{(z,y) \in (-\infty,0) \times (0,\infty)\}$	$\{(z,y) \in (-\infty,0) \times (0,\infty)\}$

With the fact that

$$0 < \int_{-\infty}^{-A_h} e^{-\frac{x^2}{2}}\, dx = \int_{A_h}^{\infty} e^{-\frac{x^2}{2}}\, dx \le e^{-\frac{A_h^2}{2}} < e^{-2}h,$$

we conclude that the midpoint scheme is asymptotically stable for any $h > 0$ as long as $\lambda + \sigma^2/2 < 0$. In other word, the midpoint scheme has the same asymptotic mean-square stable region as the equation (1.1).

2. Proof of strong convergence order for the Euler scheme

Euler scheme

$$X_{k+1} = X_k + (t_{k+1} - t_k)a(X_k) + \sigma(X_k)(W(t_{k+1}) - W(t_k)). \tag{2.1}$$

Consider a continuous version

$$\bar{X}(t) := X_k + (t - t_k)a(X_k) + \sigma(X_k)(W(t) - W(t_k)), \quad t \in [t_k, t_{k+1}). \tag{2.2}$$

$$\bar{X}(t) := X_0 + \int_0^t a(Y(s))\, ds + \int_0^t \sigma(Y(s))\, dW(s), \quad t \in [t_k, t_{k+1}). \tag{2.3}$$

where $Y(t) := X_k$ for $t \in [t_k, t_{k+1})$. Note that

- $\bar{X}(t_k) = Y(t_k) = X_k$; they coincide with the discrete solution at the gridpoints.

Now the mean-square convergence.

By the facts that a and σ are Lipschitz continuous and also by Ito's isometry,

$$\begin{aligned} \mathbb{E}[(\bar{X}(t) - X(t))^2] &= \mathbb{E}[(\int_0^t a(Y(s)) - a(X(s))\, ds + \int_0^t \sigma(Y(s)) - \sigma(X(s))\, dW(s))^2] \\ &\le C\mathbb{E}[\int_0^T \left|Y(s) - \bar{X}(s)\right|^2\, ds] + \mathbb{E}[\int_0^T \left|X(s) - \bar{X}(s)\right|^2\, ds] \end{aligned}$$

Then by Gronwall's inequality,

$$\mathbb{E}[(\bar{X}(t) - X(t))^2] \le Ce^{Ct}\mathbb{E}[\int_0^t \left|Y(s) - \bar{X}(s)\right|^2\, ds]. \tag{2.4}$$

Note that for $t \in [t_k, t_{k+1})$,

$$\left|Y(s) - \bar{X}(s)\right|^2 \le \left|a^2(X_k)\right| h^2 + \left|\sigma^2(X_k)\right| (\Delta W_k)^2 \le Ch \tag{2.5}$$

Then (with a bit more work than above) we can show that $\mathbb{E}[(\bar{X}(t) - X(t))^2] \le Ch$ and thus the Euler scheme has the mean-square convergence rate 1/2.

The weak integration of SDEs is computing the expectation

$$\mathbb{E}[f(X(T)], \tag{2.6}$$

where $f(x)$ is a sufficiently smooth function with growth at infinity not faster than a polynomial:

$$|f(x)| \le K(1 + |x|^{\kappa}) \tag{2.7}$$

for some $K > 0$ and $\kappa \ge 1$. **Weak convergence** is implied by the strong (mean-square) convergence. In fact,

$$\left|\mathbb{E}[f(\bar{X}(t_N))] - \mathbb{E}[f(X(T)]\right|^2 \le \left|\mathbb{E}[f(\bar{X}(t_N))] - \mathbb{E}[f(X(T)]\right|^2 \left|\mathbb{E}[f'(\theta\bar{X}(t_N)) + (1-\theta)X(T))]\right|^2$$

So as long as f' has at most polynomial growth at infinity, the moment exists and thus the weak convergence order is at least the same as the mean-square convergence order.

The proof of weak convergence order for the forward Euler scheme can be done by applying Ito's formula (left as an exercise).

3. Feynman-Kac formulas

Given only dynamics in the form of a SDE, we have several ways to obtain expectation of (functional of) solutions, e.g., integrating factor methods to obtain exact solutions and then expectations, moment equation methods to obtain moments of solutions. However, most SDEs do not have either explicit solutions and a closure of moment equations. One systematic technique to compute these expectations is through Feynman-Kac formula to connect with partial differential equations (PDEs). The formula can be very useful to compute expectations and conditional expectations.

3.1. **Simple forms.** Let (B_t) be a standard Brownian motion and $f:\mathbb{R}\to\mathbb{R}$ and $f\in C_b^2(\mathbb{R})$ (f and its first two derivatives are continuous, bounded). Define the function $u(t,x):\mathbb{R}^+\cup\{0\}\times\mathbb{R}\to\mathbb{R}$ by

$$u(t,x)=\mathbb{E}\big[f\big(x+B_t\big)\big].$$

We'll show that u satisfies the following partial differential equation (heat equation),

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x),\\ u(0,x) &= f(x).\end{cases}$$

Proof. Now we can use Itô's formula to prove this.

$$\begin{aligned}\partial_t u(t,x) &= \lim_{h\to 0}\frac{u(t+h,x)-u(t,x)}{h}\\ &= \lim_{h\to 0}\mathbb{E}[\frac{f(x+B_{t+h})-f(x+B_t)}{h}]\\ &= \lim_{h\to 0}\mathbb{E}[\frac{1}{2h}\int_t^{t+h}f''(x+B_s)\,ds+\frac{1}{h}\int_t^{t+h}f'(x+B_s)\,dB_s]\\ &= \lim_{h\to 0}\mathbb{E}[\frac{1}{2h}\int_t^{t+h}f''(x+B_s)\,ds],\end{aligned}$$

since $\mathbb{E}[\int_t^{t+h}f'(x+B_s)\,dB_s]=0$. In fact, we know that $\int_t^{t+h}f'(x+B_s)\,dB_s$ is a martingale because $f'(x+B_s)$ is adapted and $\mathbb{E}[\int_t^{t+h}\big(f'(x+B_s)\big)^2\,ds]<\infty$ ($f'(x+B_s)$ is bounded).

Now we proceed as

$$\begin{aligned}\partial_t u(t,x) &= \lim_{h\to 0}\mathbb{E}[\frac{1}{2h}\int_t^{t+h}f''(x+B_s)\,ds]\\ &= \mathbb{E}[\lim_{h\to 0}\frac{1}{2h}\int_t^{t+h}f''(x+B_s)\,ds]\\ &= \mathbb{E}[\lim_{h\to 0}\frac{1}{2}f''(x+B_t+(1-\theta)(B_{t+h}-B_t))],\quad 0\le\theta\le 1\quad\text{(mean value theorem)}\\ &= \frac{1}{2}\mathbb{E}[f''(x+B_t+(1-\theta)\lim_{h\to 0}(B_{t+h}-B_t)]\quad\text{(continuity of } f''\text{)}\\ &= \frac{1}{2}\mathbb{E}[f''(x+B_t)]\quad\text{(continuity of } B_t\text{)}\\ &= \frac{1}{2}\partial_x^2\big(\mathbb{E}[f(x+B_t)]\big)=\partial_x^2 u(t,x)\quad\text{(continuity)}.\end{aligned}$$

Now we prove the initial condition. By Fubini theorem and continuity of B_t, we have

$$u(0,x)=\lim_{t\to 0}\mathbb{E}[f(B_t+x)]=\mathbb{E}[f(\lim_{t\to 0}B_t+x)]=\mathbb{E}[f(x)]=f(x).$$

□

Remark 3.1. *Here we show that the expectation* $\mathbb{E}\big[f\big(x+B_t\big)\big]$ *solves the heat equation. Actually, the solution to the PDE can be represented by*

$$u(t,x)=\mathbb{E}\big[f\big(x+B_t\big)\big]=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}f(x+y)\exp(-\frac{y^2}{2t})\,dy=\int_{\mathbb{R}}f(y)p(t,x,y)\,dy,$$

where

$$p(t,x,y)=\frac{1}{\sqrt{2\pi t}}\exp(-\frac{(y-x)^2}{2t}).$$

It is required that $f(x)$ *can not growth too fast and at least we need the following condition so that* $u(t,x)$ *is well-defined as an expectation:*

$$|f(x)|\le C(1+e^{x^2}).$$

Remark 3.2. *Here the requirement* $f\in C_b^2(\mathbb{R})$ *can be relaxed. In the proof we actually use*

- *Itô's formula where we need* $f\in C^2(\mathbb{R})$ *or that* f *is convex (see Homework 7);*
- $\mathbb{E}[\int_t^{t+h} f'(B_s)\,dB_s]=0$*;*
- *mean value theorem where we need that* f'' *is continuous;*
- f *is continuous, to prove the initial condition.*

Exercise 3.3 (Feynman-Kac formula of simple forms)**.** *Denote* $X_t=x+B_t$*. Use Itô formula as in the proof above to show the following conclusions. a) Show that* $u(t,x)=\mathbb{E}[f(x+B_t)+\int_0^t\phi(s)\,ds]$ *solves*

$$\begin{cases}\frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x)+\phi(t);\\ u(0,x) &= f(x).\end{cases}$$

(Let $v=u-\int_0^s\phi(s)\,ds$ *or you can use Itô formula as in the proof above.)*

b) Show that $u(t,x)=\mathbb{E}[f(x+B_t)+\int_0^t\phi(s,x+B_s)\,ds]$ *solves*

$$\begin{cases}\frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x)+\phi(t,x);\\ u(0,x) &= f(x).\end{cases}$$

c) Show that $u(t,x)=\mathbb{E}[\exp(-\int_0^t r(t,X(s))\,ds)f(x+B_t)]$ *solves*

$$\begin{cases}\frac{\partial}{\partial t}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x)-r(t,x)u(t,x);\\ u(0,x) &= f(x).\end{cases}$$

What condition on $r(t,x)$ *do you need?*

d) What is the probabilistic representation of u *in the following equation?*

$$\begin{cases}\frac{\partial}{\partial t}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x)-r(t,x)u(t,x)+\phi(t,x), \quad x\in\mathbb{R};\\ u(0,x) &= f(x).\end{cases}$$

More examples on initial and boundary values problems are provided in Appendix.

3.2. **Generalization.** In last section, we define $X_t=x+B_t$. Now let us consider an Itô process:

$$X_t=x+\int_{t_0}^t a(X(s))\,ds+\int_{t_0}^t\sigma(\,X(s))dB_s. \tag{3.1}$$

Definition 3.4 (Characteristic operator, Infinitesimal generator)**.** *Let* $\mathcal{L}$ *be the* infinitesimal generator *of stochastic processes of* X_t*, defined by its action functions* $u:\mathbb{R}^n\to\mathbb{R}$ *and* $u\in C_0^2(\mathbb{R}^n)$

$$\mathcal{L}u(x)=\lim_{t\downarrow 0}\frac{\mathbb{E}[u(X_t)-u(x)]}{t},$$

or, equivalently,

$$\mathcal{L}u(x)=\sum_{i=1}^n a_i(x)\frac{\partial u}{\partial x_i}(x)+\frac{1}{2}\sum_{i,j=1}^n\left(\sigma\sigma^\top\right)_{i,j}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x).$$

Example 3.5. *The characteristic operator of $X_t = x + B_t$ is $\frac{1}{2}\partial_x^2$ when $x \in \mathbb{R}$. When $x \in \mathbb{R}^n$, the characteristic operator is*

$$\frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^{n} \partial_{x_i}^2.$$

Theorem 3.6 (Feynman-Kac formula)**.** *Let X_t be the solution to* (3.1)*. Let $f \in C_0^2(\mathbb{R})$ and $r(x) \in C(\mathbb{R})$ has a lower bound, then $u(t,x) = \mathbb{E}[f(X_t)\exp(-\int_0^t r(X(s))\,ds)]$ satisfies the PDE*

$$\partial_t u(t,x) = \mathcal{L}u(t,x) - r(x)u(t,x) \tag{3.2}$$

with initial condition $u(0) = f(x)$.

Proof. See Chapter 11.4 of [Kuo, 2006] or Chapter 8.2 of [Øksendal, 2003]. □

Remark 3.7. *In Chapter 8.2 of* [Øksendal, 2003]*, it is also proved that when $v(t,x)$, $\partial_t v(t,x)$, $\partial_x v(t,x)$ and $\partial_x^2 v(t,x)$ are continuous, and $v(t,x)$ is bounded when t belongs to any closed set, and v solves* (3.2) *and $v(0,x) = f(x)$, then $v(t,x) = u(t,x)$. This shows that the solution to PDE* (3.2) *can be represented by $\mathbb{E}[f(X_t)\exp(-\int_0^t r(X(s))\,ds)]$. See also Theorem 5.7.6 in Chapter 5.7.B of* [Karatzas and Shreve, 1991].

3.2.1. *Numerical methods using probalisitic representations.* If we want to solve the problem (3.2) only at a few points, we may apply the Monte-Carlo methods Observe that

$$u(t,x) = \mathbb{E}[f(X_{0,x}(t))\exp(-\int_0^t r(X_{0,x}(s))\,ds)], \quad \text{where } X_{0,x}(t) = x + \int_0^t a(X_{0,x}(s))\,ds + \int_0^t \sigma(X_{0,x}(s))\,dW(s). \tag{3.3}$$

The numerical methods can be

$$\tilde{u}(t_k, x_i) = \frac{1}{m}\sum_{j=1}^{m} f(X_k(\omega_j))\exp(-h\sum_{l=1}^{k} r(X_{k-1}(\omega_j))), \tag{3.4}$$

where

$$X_k = X_{k-1} + a(t_{k-1}, X_{k-1})h + \sigma(t_{k-1}, X_{k-1})\sqrt{h}\xi_k, \quad X_0 = x_i. \tag{3.5}$$

By the weak convergence order of the forward Euler scheme, the order is one.

3.3. **Computing conditional expectations.**

Theorem 3.8 (Feynman-Kac formula)**.** *Let X_t be the solution to* (3.1) *where a and σ satisfies certain conditions so that X_t exists and is unique. Let $f \in C_b^2(\mathbb{R})$, then $u(t,X_t) = \mathbb{E}[f(X_T)|\mathcal{F}_t] = \mathbb{E}[f(X_T)|X_t]$ satisfies the PDE*

$$\partial_t u(t,X_t) + \mathcal{L}u(t,X_t) = 0 \tag{3.6}$$

with initial condition $u(T,X_T) = f(X_T)$.

Proof. Recall that the solution X_t to (3.1) is a Markov process, see Lecture 8. Then we $\mathbb{E}[f(X_T)|\mathcal{F}_t] = \mathbb{E}[f(X_T)|X_t]$. Thus the notation $u(t,X_t) = \mathbb{E}[f(X_T)|\mathcal{F}_t]$ is meaningful. By Itô's formula,

$$u(t,X_t) = u(0,X_0) + \int_0^t \partial_x u(s,X(s))\sigma(s,X(s))\,dB_s + \int_0^t [\partial_t u(s,X(s)) + \mathcal{L}u(s,X(s))]\,ds.$$

Now show that $Y_t = \mathbb{E}[f(X_T)|\mathcal{F}_t]$ is a martingale (uniformly integrable). First of all, the conditional expected is well-defined since $f(X_t)$ is bounded. Then Y_t is a Doob's martingale and a uniformly integrable martingale. In fact, for $0 \le s \le t \le T$, we have

$$\mathbb{E}[Y_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[f(X_T)|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[f(X_T)|\mathcal{F}_s] = Y_s.$$

We know that $u(t,X_t) = Y_t$ is a martingale. Thus if

$$\int_0^t \partial_x u(s,X(s))\sigma(s,X(s))\,dB_s \tag{3.7}$$

is a martingale, we then have (WHY?)

$$\partial_t u(s, X(s)) + \mathcal{L}u(s, X(s)) = 0.$$

It remains to check that (3.7) is a martingale. Note that

$$\begin{aligned}
\int_0^t \mathbb{E}[(\partial_x u(s,X(s))\sigma(s,X(s)))^2]\,ds &= \int_0^t \mathbb{E}[\left(\mathbb{E}[\partial_x f(X_T)|\mathcal{F}_s]\sigma(s,X(s))\right)^2]\,ds \\
&= \int_0^t \mathbb{E}[\mathbb{E}[\partial_x f(X_T)\sigma(s,X(s))|\mathcal{F}_s])^2]\,ds \\
&\le \int_0^t \mathbb{E}[(\partial_x f(X_T))^2\sigma^2(s,X(s))]\,ds \text{ (Jensen's inequality)} \\
&\le \int_0^t (\partial_x f(X_T))^2\mathbb{E}[\sigma^2(s,X(s))]\,ds < \infty
\end{aligned}$$

since $f \in C_b^2(\mathbb{R})$ and $\sigma(s, X(s)) \in \mathbb{L}^2_{\mathrm{ad}}(\Omega, [0, t])$. □

Remark 3.9. *From the proof, we actually require that*

- $Y_t = \mathbb{E}[f(X_T)|\mathcal{F}_t]$ *is a martingale;*
- *we can apply Itô's formula to* $u(t, X_t)$*;*
- (3.7) *is indeed a martingale.*

This theorem is a special case of the following theorem with $r(t, x) = 0$ and $g(t, x) = 0$.

Theorem 3.10 (Feynman-Kac formula)**.** *Let* X_t *be the solution to* (3.1). *Let* $f \in C_b^2(\mathbb{R})$*, then* $u(t, X_t) = \mathbb{E}[e^{-\int_t^T r(s,X(s))\,ds} f(X_T) + \int_t^T g(s, X(s))e^{-\int_t^s r(\theta,X_\theta)\,d\theta}\,ds|\mathcal{F}_t]$ *satisfies the PDE*

$$\partial_t u(t, X_t) + \mathcal{L}u(t, X_t) = r(t, X_t)u(t, X_t) - g(t, X_t) \tag{3.8}$$

with initial condition $u(T, X_T) = f(X_T)$. *Here* $g(t, x)$ *and* $r(t, x)$ *are continuous and* $r(t, x)$ *has a lower bound.*

Proof. Apply Itô's formula to $u(t, X_t)$. □

Remark 3.11. *Black-Scholes equation is a special case of this theorem. See Appendix A.*

4. Connection to SPDEs

One starting point of studying SPDEs is to solve the nonlinear filtering problem.

4.1. Linear filtering: Kalman-Bucy (Gaussian) filter.

Consider the following linear system

$$dX = F(t)X\,dt + C(t)\,dW(t), \quad \text{(hidden state)} \tag{4.1}$$

$$dY = G(t)X\,dt + D(t)\,dB(t), \quad \text{(observation)} \tag{4.2}$$

Here $W(t)$ and $B(t)$ are independent Brownian motions.

- Combine observation and hidden dynamics
- Different confidence levels on the model and the observation data

• Recall the best linear prediction with respect to $\mathcal{F}_t^Y$ in the mean-square sense is

$$\widehat{X}(t) = \mathbb{E}[X(t)|\mathcal{F}_t^Y]. \tag{4.3}$$

• How to obtain the conditional expectation?

Innovation approach

Denote the innovation process

$$dN(t) = dY - G(t)\widehat{X}\,dt \tag{4.4}$$

Find the conditional expectation in the following steps

- Step 1. Observe that (X, Y) is a Gaussian process

- Step 2. Use the fact that $\widehat{X}(t) = \mathbb{E}[X(t)|\mathcal{F}_t^Y]$ is also a Gaussian process and thus by the definition of conditional expectation, we have

$$\widehat{X}(t) = c_0(t) + \int_0^t \tilde{f}(s)\,dY(s), \quad \tilde{f} \in L^2([0,T]). \tag{4.5}$$

- Step 3. The following process is a standard Brownian motion w.r.t $\mathcal{F}_t^Y$

$$R(t) = \int_0^t D^{-1}(s)\,dN(s) \tag{4.6}$$

Note that $\mathcal{F}_t^Y = \mathcal{F}_t^R$ and by (4.5),

$$\widehat{X}(t) = \mathbb{E}[X(t)] + \int_0^t f(s)\,dR(s), \quad f \in L^2([0,T]). \tag{4.7}$$

- Step 4. Find $f(t)$ explicitly. By the property of the conditional expectation ($\mathbb{L}^2$-projection)

$$\begin{aligned}\mathbb{E}[X(t)\int_0^t g(s)\,dR] &= \mathbb{E}[\widehat{X}(t)\int_0^t g(s)\,dR(s)] = \mathbb{E}[\int_0^t f(s)\,dR(s)\int_0^t g(s)\,dR(s)]\\ &= \int_0^t f(s)g(s)\,ds.\end{aligned} \tag{4.8}$$

Taking $g(s) = \mathbf{1}_{[0,r]}$ leads to

$$\mathbb{E}[X(t)R(t)] = \int_0^r f(s)\,ds, \quad f(s) = \partial_s \mathbb{E}[X(s)R(s)]. \tag{4.9}$$

- Step 5. Observe that $X(t)$ is solvable, as

$$X(t) = \exp(\int_0^t F(s)\,ds)X_r + \int_r^t \exp(\int_s^t F(u)\,du)C(s)\,dW(s). \tag{4.10}$$

(Step 6) Then it can be worked out that the conditional expectation satisfies a linear SDE.

Theorem 4.1. *The best mean-square prediction of $X(t)$ given Y is $\widehat{X}(t) = \mathbb{E}[X(t)|\mathcal{F}_t^Y]$ which satisfies*

$$d\widehat{X}(t) = \left(F(t) - \frac{G^2(t)S(t)}{D^2(t)}\right)\widehat{X}(t)dt + \frac{G(t)S(t)}{D^2(t)}dY(t); \quad \widehat{X}_0 = \mathbb{E}[X_0] \tag{4.11}$$

where $S(t) = \mathbb{E}[\left(X(t) - \widehat{X}(t)\right)^2]$ satisfies the (deterministic) Riccati equation

$$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)}S^2(t) + C^2(t), \quad S(0) = \mathbb{E}[(X_0 - \mathbb{E}[X_0])^2]. \tag{4.12}$$

Example 4.2 (Parameter estimation)**.**

$$dY = \theta G(t)\,dt + D(t)\,dB(t), Y(0) = 0, \tag{4.13}$$

Suppose that we have $Y(t)$, $G(t)$, $D(t)$, we want to find out what θ is.

Solution. Here $d\theta = 0$. Then

$$\frac{dS}{dt} = -\frac{G^2(t)}{D^2(t)}S^2(t), \tag{4.14}$$

Thus by Theorem 4.1,

$$\widehat{\theta}_t = \frac{\widehat{\theta}_0 S_0^{-1} + \int_0^t G(s)D^{-2}(s)dY(s)}{S_0^{-1} + \int_0^t G(s)^2 N^{-2}(s)\,ds} \tag{4.15}$$

Taking $S_0^{-1} = 0$ gives the maximum likelihood estimate.

4.2. **Nonlinear filtering.** For nonlinear problems, the above procedure cannot be applied.

$$dX = a(t,X)\,dt + \sigma(t,X)\,dW(t),\ X(0) = X_0 \quad \text{(hidden state)} \tag{4.16}$$

$$dY = h(t,X)\,dt + \sigma_0(t,Y)\,dB(t), \quad \text{(observation)} \tag{4.17}$$

The goal is $\mathbb{E}[f(X(t))|F_t^Y]$.

Change of measure

Let

$$\begin{aligned} M_t &= \exp\left\{\int_0^t \frac{\sigma_0^{-2}(s,Y(s))}{2} h^2\left(X(s)\right) ds + \int_0^t \sigma_0^{-1}(s,Y(s)) h\left(X_s\right)\, dB(s)\right\} \\ &= \exp\left\{-\int_0^t \frac{\sigma_0^{-2}(s,Y(s))}{2} h^2\left(X(s)\right) ds + \int_0^t \sigma_0^{-2}(s,Y(s)) h\left(X(s)\right) dY_s\right\}. \end{aligned}$$

Suppose that all functions are continuous and bounded. We then have

- M_t is an exponential martingale w.r.t. $\mathcal{F}_t^Y$ under the measure $\mathbb{Q}$ (Check that $\mathbb{E}^{\mathbb{Q}}[M_t] = 1$). Then $d\mathbb{Q} = M_T^{-1}\, d\mathbb{P}$ defines a new probability measure.
-

$$\mathbb{E}^{\mathbb{P}}[f(X(t))|\mathcal{F}_t^Y] = \frac{\mathbb{E}^{\mathbb{Q}}[f(X(t))M_t|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]} \tag{4.18}$$

By Ito's formula,

$$\begin{aligned} & d\mathbb{E}^{\mathbb{Q}}[f(X(t))M_t|\mathcal{F}_T^Y] \\ =\ & \mathbb{E}^{\mathbb{Q}}[d\big(f(X(t))M_t\big)|\mathcal{F}_T^Y] \\ =\ & \mathbb{E}^{\mathbb{Q}}[\mathcal{L}f(X(t))M_t|\mathcal{F}_T^Y] + \mathbb{E}^{\mathbb{Q}}[f(X(t))M_t h(t,X(t))\sigma_0^{-2}(t,Y(t))|\mathcal{F}_T^Y]\, dY(t) \end{aligned}$$

Now by the fact that $X \perp Y$ w.r.t. the measure $\mathbb{Q}$ (check that for any measurable functions f and g, $\mathbb{E}^{\mathbb{Q}}[f(X)g(Y)] = \mathbb{E}^{\mathbb{Q}}[f(X)]\mathbb{E}^{\mathbb{Q}}[g(Y)]$) and then conditioning on $\mathcal{F}_t^y$ for each conditional expectation gives

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[f(X(t))M_t|\mathcal{F}_t^Y] &= \mathbb{E}[f(X(0))] + \int_0^t \mathbb{E}^{\mathbb{Q}}[\mathcal{L}\big(f(X(s))M_s\big)|\mathcal{F}_s^Y]\, ds \\ &\quad + \int_0^t \mathbb{E}^{\mathbb{Q}}[f(X(s))h(s,X(s))\sigma_0^{-2}(s,Y(s))M_s|\mathcal{F}_s^Y]\, dY(s). \end{aligned} \tag{4.19}$$

Note that this is a SPDE (moment equations without a closure).

Denote $\varphi(t,x)$ the conditional probability and then $\mathbb{E}^{\mathbb{Q}}[f(X(t))M_t|\mathcal{F}_t^Y] = \int_{\mathbb{R}^1} f(x)\varphi(t,x)dx$. Then

$$\mathbb{E}^{\mathbb{P}}[f(X(t))|\mathcal{F}_t^Y] = \frac{\int_{\mathbb{R}^1} f(x)\varphi(t,x)dx}{\int_{\mathbb{R}^1} \varphi(t,x)dx}. \tag{4.20}$$

By the fact that $\mathbb{E}^{\mathbb{Q}}[f(X(t))M_t|\mathcal{F}_t^Y] = \int_{\mathbb{R}^1} f(x)\varphi(t,x)dx$, we have that $\varphi(t,x)$ satisfies the following Zakai equation

$$d\varphi(t,x) = \mathcal{L}^*\varphi(t,x)\,dt + h(t,x)\sigma_0^{-2}(t,Y(t))\varphi(t,x)dY(t), t > 0, x \in \mathbb{R}, \tag{4.21}$$

where $\varphi(0,x) = p(x)$ is the probability density function of X_0.

Exercise 4.3. *Show that* $\int_0^t \sigma_0^{-1}(s,Y(s))\,dY(s)$ *is a* $\mathbb{Q}$*-Brownian motion.*

Appendix A. Black-Scholes equation

Consider the SDE

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t, \quad t \geq 0.$$

Example A.1. *Suppose that* $h(x)$ *satisfies the conditions in Remark 3.9. Define that* $v(t,S_t) = \mathbb{E}[h(S_T)|\mathcal{F}_t] = \mathbb{E}[h(S_T)|S_t]$ *(by Markov property). Then by Theorem 3.8,* $v(t,x) = \mathbb{E}[h(S_T)|S_t = x]$ *satisfies the following equation*

$$\partial_t v(t,x) + rx\partial_x v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v(t,x) = 0, \quad 0 \leq t < T, \quad x \geq 0.$$

with the terminal condition $v(T,x) = h(x)$.

Further, if $S_t = 0$*, for some* $t \in [0,T]$*, then* $S_T = 0$*. This gives the boundary condition*

$$v(t,0) = h(0).$$

Example A.2. *Suppose that* $h(x)$ *satisfies the conditions in Remark 3.9. If*

$$u(t,S_t) = e^{-r(T-t)}v(t,S_t) = e^{-r(T-t)}\mathbb{E}[h(S_T)|\mathcal{F}_t] = e^{-r(T-t)}\mathbb{E}[h(S_T)|S_t],$$

then by Theorem 3.10, $u(t,x) = u(t, S_t = x) = e^{-r(T-t)}\mathbb{E}[h(S_T)|S_t = x]$ *satisfies the following partial differential equation (*Black Scholes equation*)*

$$-ru(t,x) + \partial_t u(t,x) + rx\partial_x u(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 u(t,x) = 0, \quad 0 \le t < T, \quad x > 0$$

with the terminal condition $u(T,x) = h(x)$.

Taking $h(x) = (x-K)^+$, compute $u(t,x) = e^{-r(T-t)}\mathbb{E}[(S_T - K)^+|\mathcal{F}_t]$. The terminal condition is

$$u(T,x) = (x-K)^+, \quad x \ge 0$$

and the boundary condition

$$u(t,0) = 0.$$

Exercise A.3. *Check whether the conditions in Remark 3.9 are satisfied or not. Note* $h(x) = (x-K)^+$ *is convex and continuous and thus we still can apply Itô's formula.*

Appendix B. Initial and boundary value problem

We actually showed that the connections between solutions to PDEs and Brownian motions: Itô processes connect to linear PDE with variable coefficients. Here we look at some generalizations.

- generalizations of boundary conditions (Neumann boundary conditions go with "reflected" processes).

Represent the solutions to the following equations in terms of Brownian motion.

a)

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x), \ x > 0 \\ u(0,x) &= f(x), \quad x \ge 0 \\ \frac{\partial}{\partial x}u(t,0) &= 0, \end{cases}$$

Note here $x \ge 0$ instead of $x \in \mathbb{R}$. Try to use the symmetry to write the solution in term of Brownian motion.

b)

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x), \ x > 0 \\ u(0,x) &= f(x), \ x \ge 0 \\ u(t,0) &= 0. \end{cases}$$

Try to use the symmetry to write the solution in term of Brownian motion.

c)

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x), \ x > 0 \\ u(0,x) &= f(x), \ x \ge 0 \\ u(t,0) &= g(t). \end{cases}$$

Solution to a).

$$u(t,x) = \int_0^\infty [p(t,x,y) + p(t,x,-y)]f(y)\,dy = \mathbb{E}[f(|x + B_t|)].$$

Here we use even symmetry when $x + B_t$ becomes negative, we then reflect the the value of $x + B_t$.

Solution to b). Here we extend the $f(x)$ ($x \ge 0$) in the following way $F(y) = f(y)$ when $y \ge 0$ and

$$F(y) = -f(-y), \quad y \le 0.$$

Then

$$u(t,x) = \mathbb{E}[F(x + B_t)] = \mathbb{E}[f(x + B_t)1_{\{\tau_x > t\}}],$$

where the stopping time τ_x is defined as

$$\tau_x = \tau(x,\omega) = \inf\{t > 0 | x + B_t = 0\}.$$

Solution to c). Here we extend the $f(x)$ $(x \geq 0)$ in the following way

$$f(y) = -f(-y), \quad y \leq 0.$$

Then

$$u(t,x) = \mathbb{E}[F(x+B_t)] = \mathbb{E}[f(x+B_t)1_{\tau_x \geq t}],$$

where the stopping time τ_x is defined above

$$u(t,x) = \mathbb{E}[f(x+B_t)1_{\{\tau_x > t\}}] + \mathbb{E}[g(x+B_t)1_{\{\tau_x < t\}}].$$

Exercise B.1. *Let $f = 0$. Show that*

$$u(t,x) = \mathbb{E}[g(x+B_t)1_{\{\tau_x < t\}}]$$

solve the equation in c).

Remark B.2. *We can use the equation in c) and Fourier transform to show that the density of τ_x is*

$$\frac{x}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Recall that we have computed τ_x.

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