LECTURE 2

1. Why SPDEs: motivation

Let $W(t)$ be a r-dimensional Brownian motion, i.e., $W(t) = (W_1(t), \dots, W_r(t))^\top$ where $W_i(t)$'s are mutually independent Brownian motions – Gaussian processes with the covariance function $\min(t, s)$.

Example 1.1 (Zakai equation of nonlinear filtering)**.** *Let $y(t)$ be a r-dimensional observation of a signal $x(t)$ and $y(t)$ given by*

$$y(t) \quad = \quad y_0 + \int_0^t h(x(s))\,ds + W(t),$$

where $h = (h_1, h_2, \dots, h_r)^\top$ is a $\mathbb{R}^r$-vector-valued function defined on $\mathbb{R}^d$ and the observational signal $x(t)$ satisfies the following stochastic differential equation

$$dx(t) \quad = \quad b(x(t))\,dt + \sum_{k=1}^{q} \sigma_k(x(t))\,dB_k, \quad x(0) = x_0,$$

where b and σ_k's are d-dimensional vector functions on $\mathbb{R}^d$, and $B(t) = (B_1(t),\ B_2(t) \dots, B_q(t))^\top$ is a q-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and is independent of $W(t)$. The conditional probability density function (un-normalized) of the signal $x(t)$ given $y(t)$ satisfies
(1.1)

$$du(t,x) = \frac{1}{2}\sum_{i,j=1}^{d} D_iD_j[(\sigma\sigma^\top)_{ij}u(t,x)]\,dt - \sum_{i}^{d} D_i(b_iu(t,x))\,dt + \sum_{l=1}^{r} h_lu(t,x)\,dy_l(t), \quad x \in \mathbb{R}^d.$$

Here $D_i := \partial_{x_i}$ is the partial derivative in the x_i-th direction and $\sigma^\top$ is the transpose of σ.

Example 1.2 (Pressure equation)**.** *The following model was introduced as an example of the pressure of a fluid in porous and heterogeneous (but isotropic) media at the point x over the physical domain $\mathcal{D}$ in $\mathbb{R}^d$ ($d \le 3$):*

$$-\operatorname{div}(K(x)\nabla p) = f(x), \quad p|_{\partial\mathcal{D}} = 0, \tag{1.2}$$

where $K(x) > 0$ is the permeability of media at the point x and $f(x)$ is a mass source. In a typical porous media, $K(x)$ is fluctuating in a unpredictable and irregular way and can be modeled with a stochastic process.

Example 1.3 (Turbulence model)**.** *The stochastic incompressible Navier-Stokes equation reads*

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \sigma(\mathbf{u})\dot{W}^Q, \quad \operatorname{div}\mathbf{u} = 0, \tag{1.3}$$

where σ is Lipschitz continuous over the physical domain in $\mathbb{R}^d$ ($d = 2, 3$). Here $\mathbb{E}[W^Q(x,t)W^Q(y,s)] = q(x,y)\min(s,t)$ and $q(x,x)$ is square-integrable over the physical domain.

Example 1.4 (Reaction-diffusion equation)**.**

$$du = \nu\Delta u + f(u) + \sigma(u)\dot{W}^Q(t,x).$$

This may represent a wide class of SPDEs:

- *In materials, the stochastic Allen-Cahn equation, where $f(u) = u(1-u)(1+u)$, $\sigma(u)$ is a constant and $\dot{W}^Q(t,x)$ is space-time white noise.*

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- *In population genetics, this equation has been used to model changes in the structure of population in time and space, where* $\dot{W}^Q(t,x)$ *is a Gaussian process white in time but color in space. For example,* $f(u) = 0$ $\sigma(u) = \gamma\sqrt{\max(u,0)}$, *where* γ *is a constant and* u *is the mass distribution of the population. Also,* $f(u) = \alpha u - \beta$, $\sigma(u) = \gamma\sqrt{\max(u(1-u),0)}$, *where* α, β, γ *are constants.*
- *When* $\sigma(u)$ *is a small constant, the equation can be treated as random perturbation of deterministic equations (*$\sigma(u) = 0$*).*

In the first and third cases, we say that the equation has an additive noise *as the coefficients of noise are independent of the solution. In the second case, we say that the equation has a* multiplicative noise *since the coefficient of noise depends on the solution itself.*

2. How to obtain a series expansion of a stochastic process

When the kernel $K(x,y)$ is positive definite, then we can apply KL decomposition.

Theorem 2.1 (Karhunen-Loève expansion)**.** *Let* $X(t)$ *be a Gaussian stochastic process defined over a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $t \in [a,b]$, $-\infty < a < b < \infty$. *Suppose that* $X(t)$ *has a continuous covariance function* $C(t,s) = \mathbb{C}ov[(X(t), X(s))] = \mathbb{E}[(X(t) - \mathbb{E}[X(t)])(X(s) - \mathbb{E}[X(s)])]$. *Then* $X(t)$ *admits the following representation*

$$X(t) = \mathbb{E}[X(t)] + \sum_{k=1}^{\infty} Z_k e_k(t),$$

where the convergence is in L^2, *uniform in* t *(i.e.* $\lim_{n\to\infty} \max_{t\in[a,b]} \mathbb{E}[(X(t) - \mathbb{E}[X(t)] - \sum_{k=1}^{n} Z_k e_k(t))^2] = 0$*) and*

$$Z_k = \int_a^b \big(X(t) - \mathbb{E}[X(t)]\big) e_k(t)\, dt.$$

Here the eigenfunctions e_k*'s of* C_X *with respective eigenvalues* λ_k*'s form an orthonormal basis of* $L^2([a,b])$ *and*

$$\int_a^b C(t,s) e_k(t)\, dt = \lambda_k e_k(s), \quad k \geq 1.$$

Furthermore, the random variables Z_k*'s have zero-mean, are uncorrelated and have variance* λ_k

$$\mathbb{E}[Z_k] = 0, \quad \textit{for all } k \geq 1 \qquad \textit{and} \qquad \mathbb{E}[Z_i Z_j] = \delta_{ij}\lambda_j, \quad \textit{for all } i,j \geq 1.$$

The stochastic process $X(t)$ can be non-Gaussian.

The covariance function $C(t,s)$ can be represented as $C(t,s) = \sum_{k=1}^{\infty} \lambda_k e_k(t) e_k(s)$. The variance of $X(t)$ is the sum of the variances of the individual components of the sum:

$$\mathbb{V}\mathrm{ar}[X(t)] = \mathbb{E}[(X(t) - \mathbb{E}[X(t)])^2] = \sum_{k=0}^{\infty} e_k^2(t) \mathbb{V}\mathrm{ar}[Z_k] = \sum_{k=1}^{\infty} \lambda_k e_k^2(t).$$

Here Z_k are uncorrelated random variables.

The domain where the process is defined can be extended to bounded domains in $\mathbb{R}^d$.

Example 2.2 (Brownian motion)**.** *When* $C(t,s) = \min(t,s)$, $t \in [0,1]$, *then the Gaussian process* $X(t)$ *can be written as*

$$X(t) = \sqrt{2} \sum_{i=1}^{\infty} \xi_k \frac{\sin\left(\left(k - \frac{1}{2}\right)\pi t\right)}{\left(k - \frac{1}{2}\right)\pi}.$$

Here ξ_k's are mutually independent standard Gaussian random variables. One can show that for $t, s \in [0,1]$, the eigenvectors of the covariance function $\min(t,s)$ *are*

$$e_k(t) = \sqrt{2}\sin\left(\left(k - \tfrac{1}{2}\right)\pi t\right),$$

and the corresponding eigenvalues are

$$\lambda_k = \frac{1}{(k-\frac{1}{2})^2\pi^2}.$$

Example 2.3 (Brownian Bridge)**.** *Let $X(t)$, $0 \le t \le 1$, be the Gaussian process in Example 2.2. Then $Y(t) = X(t) - tX(1)$, $0 \le t \le 1$, is also a Gaussian process and admits the following Karhunen-Loève expansion:*

$$Y(t) = \sum_{k=1}^{\infty} \eta_k \frac{\sqrt{2}\sin(k\pi t)}{k\pi}.$$

Here η_k's are mutually independent standard Gaussian random variables.

Example 2.4 (Ornstein-Uhlenbeck process)**.** *Consider a centered one-dimensional Gaussian process with an exponential covariance function* $\exp(-\frac{|t-s|}{l})$*. The Karhunen-Loève expansion of such a Gaussian process over $[-a,a]$ is*

$$O(t) = \sum_{k=1}^{\infty} \xi_k \sqrt{\lambda_k} e_k(t),$$

where $\lambda_k = \frac{2l}{l^2\theta_k^2+1}$ and the corresponding eigenvalues are

$$e_{2i}(t) = \frac{\cos(\theta_{2i}t)}{\sqrt{2 + \frac{\sin(2\theta_{2i}a)}{2\theta_{2i}}}}, \quad e_{2i-1}(t) = \frac{\sin(\theta_{2i-1}t)}{\sqrt{2 - \frac{\sin(2\theta_{2i-1}a)}{2\theta_{2i-1}}}}, \quad \text{for all } i \ge 1,\ t \in [-a,a].$$

The θ_{2k-1}'s are solutions to the following transcendental equation

$$1 - l\theta\tan(a\theta) = 0.$$

The θ_{2k}'s are solutions to the following transcendental equation

$$l\theta + \tan(a\theta) = 0.$$

Numerical Consideration. The Karhunen-Loève expansion can be found numerically, and in practice only a finite number of terms in the expansion are required. Specifically, we usually perform a principal component analysis by truncating the sum at some N such that

$$\frac{\sum_{i=1}^{N}\lambda_i}{\sum_{i=1}^{\infty}\lambda_i} = \frac{\sum_{i=1}^{N}\lambda_i}{\int_a^b \mathbb{V}\mathrm{ar}[X(t)]\,dt} \ge \alpha.$$

Here α is typically taken as 0.9, 0.95 and 0.99. The eigenvalues and eigenfunctions are found by solving numerically the following eigenproblem (integral equation):

$$\int_a^b C(t,s)e_k(t)\,ds = \lambda_k e_k(s), \quad s \in [a,b] \text{ and } k = 1,2,\ldots,N.$$

2.1. **Convergence of random series.**

Theorem 2.5 (Kolmogrov Three-series Theorem)**.** *Let X_n be a sequence of independent RVs. Define that $X_n^{(k)} = X_n 1_{\{|X_n|\le k\}}$ for any $k \ge 0$. Then $\sum_{j=1}^{\infty} X_n$ converges if and only if*

(1) $\sum_{n=1}^{\infty}\mathbb{P}(|X_n| \ge k) < \infty$ *and*
(2) $\sum_{n=1}^{\infty}\mathbb{E}[X_n^{(k)}] < \infty$ *a.s. and*
(3) $\sum_{n=1}^{\infty}\mathbb{V}ar[X_n^{(k)}] < \infty$ *a.s.* .

We need the following lemma to prove Kolmogrov Three-series Theorem.

Lemma 2.6. *Suppose that* (X_n) *independent and* $|X_n| \le K$ *for some* $K > 0$*. Then* $\sum_{j=1}^{\infty} X_n$ *converges a.s. if and only if* $\sum_{j=1}^{\infty} \mathbb{E}[X_n]$ *converges and* $\sum_{j=1}^{\infty} \mathbb{V}ar[X_n] < \infty$.

Proof. ($\Longleftarrow$). $\sum_{j=1}^{\infty} \mathbb{V}\mathrm{ar}[X_n] < \infty$ leads to $\sum_{j=1}^{\infty}(X_n - \mathbb{E}[X_n]) < \infty$ a.s. . The fact that $\sum_{j=1}^{\infty} \mathbb{E}[X_n]$ converges then gives that $\sum_{j=1}^{\infty} X_n$ converges.

($\Longrightarrow$). See proof at [Chapter 12.4] of[Williams, 1991]. □

Exercise 2.7. $\{X_n\}$ *are i.i.d. and* $\mathbb{P}(X_n = \pm 1) = \frac{1}{2}$*. Let* a_n *be real numbers. Then*

$$\sum_{n=1}^{\infty} a_n X_n \text{ converges a.s. if and only if } \sum_{n=1}^{\infty} a_n^2 < \infty.$$

Remark 2.8. *Compare with the following deterministic series. The following alternating sequence converges:*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = (\sqrt{2}-1)\zeta(\frac{1}{2}), \text{ where } \zeta(x) \text{ is the Riemann Zeta function.)}$$

However, its squared summation which is harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

3. Pathwise smoothness of a stochastic process

Definition 3.1. *Consider two stochastic processes,* $X(t)$ *and* $Y(t)$*, defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. We call* $Y(t)$ *a modification (or version) of* $X(t)$ *if for every* $t \ge 0$ *, we have*

$$\mathbb{P}(X(t) = Y(t)) = 1.$$

Definition 3.2 (mean-square continuity)**.** *A stochastic processes* $X(t)$ *is mean-square continuous if for every* $t \ge 0$ *, we have*

$$\lim_{h\to 0} \mathbb{E}[|X(t) - X(t+h)|^2] = 0.$$

Theorem 3.3 (Kolomogrov and Centsov continuity theorem)**.** *Given a stochastic process* $X(t)$ *with* $t \in [a, b]$*, if there exist constants* $p > r$*,* $K > 0$ *such that*

$$\mathbb{E}[|X(t) - X(s)|^p] \le K\,|t-s|^{1+r}, \text{ for } t, s \in [a, b],$$

then $X(t)$ *has a modification* $Y(t)$ *which is almost everywhere (in* ω*) continuous: for all* $t, s \in [a, b]$*,*

$$|Y(t,\omega) - Y(s,\omega)| \le C(\omega)\,|t-s|^{\alpha},\ 0 < \alpha < \frac{r}{p}.$$

For $X(\boldsymbol{t})$, $\boldsymbol{t} \in T \subseteq \mathbb{R}^d$, if there exist constants $p > r$, K such that

$$\mathbb{E}[|X(\boldsymbol{t}) - X(\boldsymbol{s})|^p] \le K\,|\boldsymbol{t}-\boldsymbol{s}|^{d+r}, \text{ for } \boldsymbol{t}, \boldsymbol{s} \in T,$$

then $X(\boldsymbol{t})$ has a modification $Y(\boldsymbol{t})$ which is almost everywhere in ω continuous: for all $\boldsymbol{t}$, $\boldsymbol{s} \in T$,

$$\mathbb{E}[\big(\sup_{\boldsymbol{s}\neq\boldsymbol{t}} \frac{|Y(\boldsymbol{t},\omega) - Y(\boldsymbol{s},\omega)|}{|\boldsymbol{t}-\boldsymbol{s}|}^{\alpha}\big)^p] < \infty,\ 0 < \alpha < \frac{r}{p}.$$

For a proof, see `http://individual.utoronto.ca/jordanbell/notes/kolmogorovcontinuity.pdf` or Pages 271-275 of [Walsh, 1986].

Exercise 3.4. *Let* $\xi \sim \mathcal{N}(\mu, \sigma^2)$*,* $\sigma > 0$*. Then* $\mathbb{E}[|\xi|^p] = \sigma^p \frac{2^{p/2}\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}$.

For Gaussian process (mean 0 for simplicity), it is straightforward to check the mean-square continuity.

$$\mathbb{E}[|X(t)-X(s)|^p] \le C_p\big(\mathbb{E}[|X(t)-X(s)|^2]\big)^{\frac{p}{2}} = C_p(K(t,t)+K(s,s)-2K(s,t))^{\frac{p}{2}}.$$

Some special cases:

- $K(t,s)=t\wedge s$. $\mathbb{E}[|X(t)-X(s)|^p] \le C_p\,|t-s|^{\frac{p}{2}}$.
- $K(t,s)=\exp(-|t-s|)$, $\mathbb{E}[|X(t)-X(s)|^p] \le C_p\,|2-\exp(-|t-s|)|^{\frac{p}{2}} \le C_p\,|t-s|^{\frac{p}{2}}$.

Both are only Hölder continuous with exponent $\gamma < 1/2$.

Exercise 3.5. *What is the pathwise smoothness of a Gaussian process with kernel* $\exp(-|x-y|^2)$*?*

Definition 3.6 (mean-square derivatives)**.** *A stochastic processes $X(t)$ is mean-square differentiable if for every $t\ge 0$, we have*

$$\lim_{h\to 0}\mathbb{E}[\left|\frac{X(t+h)-X(t)}{h}-\frac{d}{dt}X(t)\right|^2]=0.$$

4. Brownian motion

Definition 4.1 (One dimensional Brownian motion)**.** *A one-dimensional continuous time stochastic process $W(t)$ is called a standard Brownian motion if*

- *$W(t)$ is almost surely continuous in t,*
- *$W(t)$ has independent increments,*
- *$W(t)-W(s)$ obeys the normal distribution with mean zero and variance $t-s$.*
- *$W(0)=0$.*

It can be readily shown that $W(t)$ is Gaussian process. We then call $\dot{W}(t)=\frac{d}{dt}W$, formally the first-order derivative of $W(t)$ in time, *white noise*.

By Example 2.2, then the Brownian motion $W(t)$, $t\in[0,1]$ can be represented by

$$W(t)=\sqrt{2}\sum_{i=1}^{\infty}\xi_i\frac{\sin\left(\left(i-\frac{1}{2}\right)\pi t\right)}{\left(i-\frac{1}{2}\right)\pi},\quad t\in[0,1],$$

where ξ_i's are mutually independent standard Gaussian random variables. The Brownian motion and white noise can also be defined in terms of orthogonal expansions. Suppose that $\{m_k(t)\}_{k\ge 1}$ is a complete orthonormal system (CONS) in $L^2([0,T])$. The Brownian motion $W(t)$, $t\in[0,T]$ can be defined by

$$W(t)=\sum_{i=1}^{\infty}\xi_i\int_0^t m_i(s)\,ds,\ t\in[0,T], \tag{4.1}$$

where ξ_i's are mutually independent standard Gaussian random variables. It can be checked that the Gaussian process defined by (4.1) is indeed a standard Brownian motion by Definition 4.1. Correspondingly, the white noise is defined by

$$\dot{W}(t)=\sum_{i=1}^{\infty}\xi_i m_i(t),\ t\in[0,T]. \tag{4.2}$$

When $m_i(t)=\sqrt{2/T}\cos((i-1/2)\pi t/T)$, $i\ge 1$, then the representation (4.1) coincides with the *Karhunen-Loève expansion* of Brownian motion in Example 2.2 when $T=1$.

Definition 4.2 (Multidimensional Brownian motion)**.** *A continuous stochastic process $W_t=(W_1(t),\dots,W_m(t))^\top$ is called an m-dimensional Brownian motion on $\mathbb{R}^m$ when $W_i(t)$ are mutually independent standard Brownian motions on $\mathbb{R}$.*

Definition 4.3 (Multidimensional Brownian motion, alternative definitions)**.** *An $\mathbb{R}^d$-valued continuous Gaussian process $X(t)$ with mean function $\mu(t) = \mathbb{E}[X(t)]$ and the covariance function $C(t,s) = \mathbb{C}ov[(X(t), X(s))] = \mathbb{E}[(X(s)-\mu(s))(X(t)-\mu(t))^\top]$ is called a d-dimensional Brownian motion if for any $0 \le t_0 < t_1 < \cdots < t_n$,*

- *$X(t_i)$ and $X(t_{i+1}) - X(t_i)$ are independent;*
- *the covariance function (a matrix) is a diagonal matrix with entries $\min(t_i, t_j)$, $0 \le i, j \le n$.*

When $\mu(t) = 0$ for all t and $C(t,s) = \min(t,s)$, the Gaussian process is called a standard Brownian motion.

4.1. **Some properties of Brownian motion.**

Theorem 4.4. *The covariance $\mathbb{C}ov[(W(t), W(s))] = \mathbb{E}[W(t)W(s)] = \min(t,s)$.*

- *Time-homogeneity: For any $s > 0$, $\tilde{W}(t) = W(t+s) - W(s)$ is a Brownian motion, independent of $\sigma(W(u), u \le s)$.*
- *Brownian scaling: For every $c > 0$, $cW(t/c^2)$ is a Brownian motion.*
- *Time inversion: Let $\tilde{W}(0) = 0$ and $\tilde{W}(t) = tW(1/t)$, $t > 0$. Then $\tilde{W}(t)$ is a Brownian motion.*

Corollary 4.5 (Strong law of large numbers for Brownian motion)**.** *If $W(t)$ is a Brownian motion, then it holds almost surely that*

$$\lim_{t\to\infty} \frac{W(t)}{t} = 0.$$

Theorem 4.6 (Law of the iterated logarithm)**.** *Let W_t be a standard Brownian motion. Then*

$$\mathbb{P}(\limsup_{t\to 0} \frac{W_t}{\sqrt{2t\,|\log\log(t)|}} = 1) = 1, \quad \mathbb{P}(\liminf_{t\to 0} \frac{W_t}{\sqrt{2t\,|\log\log(t)|}} = -1) = 1.$$

$$\mathbb{P}(\limsup_{t\to \infty} \frac{W_t}{\sqrt{2t\log\log(t)}} = 1) = 1, \quad \mathbb{P}(\liminf_{t\to \infty} \frac{W_t}{\sqrt{2t\log\log(t)}} = -1) = 1.$$

See `https://www.math.unl.edu/~sdunbar1/ProbabilityTheory/Lessons/BernoulliTrials/IteratedLogarithm/iteratedlogarithm.pdf` for a proof.

Example 4.7 (Ornstein-Uhlenbeck process)**.** *Consider a centered one-dimensional Gaussian process with exponential covariance function $\exp(-\frac{|t-s|}{\sigma})$. The Gaussian process is usually called a Ornstein-Uhlenbeck process. Suppose that $W(t)$ is a standard Brownian motion. For $t \ge 0$, the Ornstein-Uhlenbeck process can be written as*

$$O(t) = e^{-\frac{t}{\sigma}} W(e^{\frac{2t}{\sigma}}).$$

Example 4.8. *The Brownian bridge $X(t)$ is a one-dimensional Gaussian process with time $t \in [0,1]$ and covariance $\mathbb{C}ov[(X(t), X(s))] = \min(t,s) - ts = \begin{cases} s(1-t), & 0 \le s \le t \le 1. \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$*

Suppose that $W(t)$ is a standard Brownian motion. Then $X(t)$ can be represented by

$$X(t) = W(t) - tW(1) = t\big(W(t) - W(1)\big) + (1-t)\big(W(t) - W(0)\big), \quad 0 \le t \le 1.$$

The process $X(t)$ bridges $W(t) - W(1)$ and $W(t) - W(0)$. It can be readily verified that $\mathbb{C}\mathrm{ov}[(X(t), X(s))] = \min(t,s) - ts$ and $X(t)$ is continuous and starting from 0. Moreover,

$$W(t) = (t+1)X(\frac{t}{t+1}), \quad X(t) = (1-t)W(\frac{t}{1-t}).$$

Theorem 4.9. *For $\alpha < \frac{1}{2}$, the Brownian motion has a modification which is of locally Hölder continuous of order α.*

Proof. For integer $n \geq 1$, by Kolomogrov and Centsov continuity theorem, it only requires to show that

$$\mathbb{E}[|W(t) - W(s)|^{2n}] \leq C_n |t-s|^n .$$

. □

Theorem 4.10 ([Karatzas and Shreve, 1991, Section 2.9.D]). *The Brownian motion is nowhere differentiable: for almost all ω, the sample path (realization, trajectory) $W(t,\omega)$ is nowhere differentiable as function of t. Moreover, for almost all ω, the path (realization, trajectory) $W(t,\omega)$ is nowhere Hölder continuous with exponent $\alpha > \frac{1}{2}$.*

Definition 4.11 (p-variation). *The p-variation of a real-valued function f, defined on an interval $[a,b] \subset \mathbb{R}$, is the quantity*

$$|f|_{p,\mathrm{TV}} = \sup_{\Pi_n,\, |\Pi_n| \to 0} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|^p,$$

where the supremum runs over the set of all partitions Π_n of the given interval.

Theorem 4.12 (unbounded total variation of Brownian motion). *The paths (realizations, trajectories) of Brownian motion are of infinite total variation almost surely (a.s., with probability one).*

Proof. Without loss of generality, let's consider the interval $[0,1]$.

$$|W|_{1,\mathrm{TV}} = \sup_{\Pi_n} \sum_{i=0}^{n-1} |W(t_{i+1}) - W(t_i)| \geq \sum_{i=0}^{n-1} |W(\frac{i+1}{n}) - W(\frac{i}{n})| =: V_n.$$

Denote by $W(\frac{i+1}{n}) - W(\frac{i}{n}) = \frac{\xi_i}{\sqrt{n}}$. Then ξ_i's are i.i.d. $\mathcal{N}(0,1)$ random variables. Observe that $\mathbb{E}[V_n] = \sqrt{n}\mathbb{E}[|\xi_1|]$ and $\mathbb{V}\mathrm{ar}[V_n] = 1 - (\mathbb{E}[|\xi_1|])^2$. Then it follows from the Chebyshev inequality, we have

$$\begin{aligned}
\mathbb{P}(V_n \geq \frac{1}{2}\mathbb{E}[|\xi_1|]\sqrt{n}) &= \mathbb{P}(V_n - \mathbb{E}[|\xi_1|]\sqrt{n} \geq -\frac{1}{2}\mathbb{E}[|\xi_1|]\sqrt{n}) \\
&\geq 1 - \mathbb{P}(\left|V_n - \mathbb{E}[|\xi_1|]\sqrt{n}\right| \geq \frac{1}{2}\mathbb{E}[|\xi_1|]\sqrt{n}) \\
&\geq 1 - \frac{\mathbb{V}\mathrm{ar}[V_n]}{\left(\frac{1}{2}\mathbb{E}[|\xi_1|]\sqrt{n}\right)^2} = 1 - 4\frac{1 - (\mathbb{E}[|\xi_1|])^2}{n(\mathbb{E}[|\xi_1|])^2}.
\end{aligned}$$

Thus we have

$$\mathbb{P}(|W|_{1,\mathrm{TV}} \geq \frac{\mathbb{E}[|\xi_1|]}{2}\sqrt{n}) \geq \mathbb{P}(V_n \geq \frac{\mathbb{E}[|\xi_1|]}{2}\sqrt{n}) = 1 - 4\frac{1 - (\mathbb{E}[|\xi_1|])^2}{n(\mathbb{E}[|\xi_1|])^2}.$$

Letting $n \to \infty$, we obtain

$$\mathbb{P}(|W|_{1,\mathrm{TV}} = \infty) = 1.$$

□

References

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