LECTURE 11 STOCHASTIC HYPERBOLIC EQUATIONS AND THEIR NUMERICAL METHODS

1. First-order wave equations

Example 1.1 (Advection-reaction)**.**

$$\partial_t u + c\partial_x u = \sigma(u-1)\circ \dot{W}(x), \qquad x\in[0,L], c>0 \tag{1.1}$$

with initial condition $u_0(x)$ *and zero inflow. The stochastic product* $u\circ\dot{W}$ *is the Stratonovich product.*

Exercise 1.2. *Which boundary should the inflow be given at?*

The exact solution of (1.1) is

$$u = 1 + [u_0(x-ct)-1]\exp[\sigma W(x) - \sigma W(x-ct)]. \tag{1.2}$$

Equation (1.1) can be written in Ito's form

$$\partial_t u + \partial_x u = \frac{\sigma^2}{2}(u-1) + \sigma(u-1)\diamond W(x), \qquad x\in[0,L], \tag{1.3}$$

where '$\diamond$' represents the Ito-Wick product. Applying the truncated spectral expansion $W_{\mathsf{n}} = \sum_{k=1}^{\mathsf{n}} m_k(x)\xi_k$ in one-dimensional physical space, we then have the following approximation to Equation (1.1):

$$\partial_t u_{\mathsf{n}} + c\partial_x u_{\mathsf{n}} = \sigma(u_{\mathsf{n}}-1)\frac{d}{dx}W_{\mathsf{n}}(x), \qquad x\in[0,L] \tag{1.4}$$

whose solution is

$$u_{\mathsf{n}} = 1 + [u_0(x-ct)-1]\exp[\sigma W_{\mathsf{n}}(x) - \sigma W_{\mathsf{n}}(x-ct)]. \tag{1.5}$$

Theorem 1.3. *Let* u *be the solution to* (1.1) *and* u_{n} *the solution to* (1.4)*. Then we have*

$$\mathbb{E}[|u-u_{\mathsf{n}}|^2] \le C_1\frac{1}{\mathsf{n}}, \quad \left|\mathbb{E}[u^k - u_{\mathsf{n}}^k]\right| \le C_2\frac{1}{\mathsf{n}}, \quad \forall k>0, \tag{1.6}$$

where C_1 *and* C_2 *depend only on* t*,* x *and* σ *in the former inequality and* C_2 *depends also on* k *in the latter.*

Remark 1.4. *For one-dimensional advection equations with multiplicative noise, we have the order of* $\frac{1}{\sqrt{\mathsf{n}}}$ *for strong convergence and* $\frac{1}{\mathsf{n}}$ *for weak convergence. We do not expect better convergence order as in the case of elliptic equation, where the smoothing of the inverse of Laplacian operator is involved; see Lecture 7 for a comparison.*

Full discretization. We can use the upwind scheme in x and the forward Euler scheme in t to solve this problem.

$$u_j^{n+1} - u_j^n + c\frac{\delta t}{\delta x}(u_j^n - u_{j-1}^n) = \frac{\sigma^2\delta t}{2}(u_j^n - 1) + \sigma(u_j^n-1)\Delta W_n.$$

When $\sigma=0$,

$$u_j^{n+1} - u_j^n + c\frac{\delta t}{\delta x}(u_j^n - u_{j-1}^n) = 0.$$

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The stability condition is $c\frac{\delta t}{\delta x} \le 1$. When $\frac{\delta t}{\delta x} = 1$, $u_j^{n+1} = u_{j-1}^n$, this matches the characteristic of the equation $u_t + cu_x = 0$, for which the solution is $u_0(x - ct)$.

Numerical diffusion and dissipation of a scheme can be analyzed using the *modified equation method*. In this method, we will find a PDE which the exact solution of the discretized equation satisfies and this PDE is generally different from the one we want to solve. Specifically, we expand all nodal values in the difference scheme in a double Taylor series about a single point (x_j, t_n) of the space-time mesh to obtain a PDE with the explicitly-written coefficients of u_{xx} (diffusion) and u_{xxx} (dissipation). The modified equation is

$$\partial_t u + c\partial_x u = \underbrace{c\frac{\delta x}{2}(1-\rho)\partial_x^2 u}_{\text{numerical diffusion}} + \underbrace{c\frac{(\delta x)^2}{6}(3\rho - 2\rho^2 - 1)\partial_x^3 u}_{\text{numerical dispersion}} + O(\delta t, \delta x), \quad \rho = c\frac{\delta t}{\delta x}. \tag{1.7}$$

The derivation of this equation is provided in the appendix. From the modified equation, we observe that

- When $\rho = 1$, there is no numerical diffusion or dispersion.
- When $\rho > 1$, we will have anti-diffusion, which leads to instability. In fact, we need $\rho \le 1$ from a stability analysis.
- The scheme is of first-order convergence.

Example 1.5 (Advection)**.**

$$\partial_t u + \sigma\partial_x u \circ \dot{W}(t) = 0, \qquad x \in [0, L] \tag{1.8}$$

with initial condition $u_0(x)$.

The solution is $u = u_0(x - \sigma w(t))$. In Ito's form,

$$\partial_t u + \sigma\partial_x u\dot{W}(t) = \frac{\sigma^2}{2}\partial_x^2 u, \qquad x \in [0, L]. \tag{1.9}$$

Remark 1.6. *Here if a boundary condition (deterministic) should be given, it has to be periodic boundary condition as the characteristic is random and changes its sign randomly.*

We then use the following scheme (upwind plus the explicit Euler for diffusion)

$$u_j^{n+1} - u_j^n + \sigma\frac{\delta t}{\delta x}(u_j^n - u_{j-1}^n)\frac{\Delta W_n}{\delta t} = \frac{\sigma^2}{2}\delta t\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}. \tag{1.10}$$

Phase speed error analysis (for spatial discretizations). Let's consider a continuous time model with initial condition e^{ikx} ($L = 2\pi$ for simplicity). (Here we mimic the finite difference scheme in x of the upwind.)

$$\partial_t v + \sigma\frac{v(t,x) - v(t, x - \delta x)}{\delta x} \circ \dot{W}(t) = 0. \tag{1.11}$$

We are looking for a solution of the following form $v = e^{ik(x - c(t))}$. Plugging this form into the equation above, we have

$$-ikc'(t) + \sigma\frac{1 - e^{ik(x-\delta x)}}{\delta x} \circ \dot{W}(t) = 0,$$

Then $c(t) = \sigma\frac{1-e^{ik(x-\delta x)}}{ik\delta x}W(t) = \sigma W(t) + \sigma\frac{ik\delta x}{2}W(t)$. So the phase speed of the upwind scheme has an error of order $k\delta x W(t)$, which implied that for large k and t, we may not have practical step size δx to keep the phase speed. We will need a high-order method, which may allows a smaller phase speed error for relative large k and/or t.

Exercise 1.7. *What is the mean-square stability condition for the scheme* (1.10)*?*

Remark 1.8. *It might be more practical to consider the following model, instead of* (1.8),

$$\partial_t u + \partial_x u \circ (c + \sigma \dot{W}(t)) = 0, \qquad x \in [0, L] \tag{1.12}$$

where σ is small compared to c.

2. Second-order wave equations

Consider the following problem on a bounded domain with certain boundary conditions (we assume them vanishing for simplicity).

$$u_{tt} = u_{xx} + f(t, x), \quad t > 0,\, x \in \mathcal{D} = (0, L). \tag{2.1}$$

The solution of this equation can be written as

$$u = \int_{\mathcal{D}} \partial_t K(x, y; t) u_0(y)\, dy + \int_{\mathcal{D}} K(x, y; t) v_0(y)\, dy + \int_0^t \int_{\mathcal{D}} K(x, y.t - s) f(s, y)\, dy\, ds. \tag{2.2}$$

where

$$K(x, y; t) = \sum_{k=1}^{\infty} \frac{\sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} e_k(x) e_k(y).$$

Here λ_k, e_k are the eigenpairs of the problem $-u_{xx} = \lambda u$ with the given boundary condition.

When $f = \dot{W}^Q(t, x)$ and $u_0 = v_0 = 0$, the solution is then

$$u = \int_0^t \int_{\mathcal{D}} K(x, y; t - s)\, dW^Q(s, y). \tag{2.3}$$

If Q has e_k's as their eigfunctions (which is always the case if $Q = I$), we write $\dot{W}^Q(t, x) = \sum_{k=1}^{\infty} \sqrt{q_k} e_k(x) \dot{W}_k(t)$ and thus

$$u = \sum_{k=1}^{\infty} \sqrt{q_k} \int_0^t \frac{\sin(\sqrt{\lambda_k}(t - s))}{\sqrt{\lambda_k}}\, dW_k(s).$$

2.1. Linear equations on bounded domains in 1D.

Consider the following wave equation

$$u_{tt}(t, x) = u_{xx}(t, x) + \dot{W}(t, x), t > 0, x \in (0, \pi) \tag{2.4}$$

with vanishing initial and boundary conditions: $u(0, x) = u_t(0, x) = u(t, 0) = u(t, \pi) = 0$. Then

$$u(t, x) = \sum_{k=1}^{\infty} e_k(x) u_k(t), \quad e_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx).$$

Then we have

$$\ddot{u}_k = -k^2 u_k + \dot{w}_k, \quad u_k(0) = \dot{u}_k(0) = 0. \tag{2.5}$$

It can be readily shown that

$$u_k(t) = \frac{1}{k} \int_0^t \sin(k(t - s))\, dw_k(s).$$

Then

$$\mathbb{E}[u^2(t, x)] = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} (\frac{t}{2} - \frac{\sin(2kt)}{4k}) \sin^2(kx). \tag{2.6}$$

Exercise 2.1. *What is the regularity of the solution in t and x? (Hölder continuous $1/2 - \varepsilon$).*

2.2. **Linear equations on $\mathbb{R}$.** If the equation (2.1) is considered on the whole space, it is convenient to applythe characteristic decomposition. Let

$$v = \int_0^t \partial_x u\, ds + \int_0^x f(y)\, dy. \tag{2.7}$$

Then we have

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

Then apply the eigenvalue decomposition to the matrix A. Denote $A = S\Lambda S^{-1}$. Let $\binom{\tilde u}{\tilde v} = S^{-1}\binom{u}{v}$, where $S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and we obtain that

$$\partial_t \begin{pmatrix} \tilde u \\ \tilde v \end{pmatrix} = \Lambda \begin{pmatrix} \tilde u \\ \tilde v \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.9}$$

We then have a separable uncoupled equations to solve. However, the most important thing is to have the matrix S^{-1}, which describes the characteristic of the wave equation.

Consider the stochastic wave equation in $\mathbb{R}$ driven by the space-time white noise

$$u_{tt}(t,x) = u_{xx}(t,x) + \dot W(t,x), t > 0, x \in \mathbb{R}, u(0,x) = u_t(0,x) = 0 \tag{2.10}$$

Let $(\xi, \eta)^\top = S^{-1}(t,x)^\top$ or in an equivalent form

$$\xi = (t+x)/\sqrt{2}, \quad \eta = (t-x)/\sqrt{2}. \tag{2.11}$$

We are led to

$$2\frac{\partial^2 \widetilde u}{\partial \xi \partial \eta} = \dot{\widetilde W}(\eta, \xi), \qquad \text{where } \widetilde u(\eta,\xi) = u(t,x), \quad \dot{\widetilde W}(\eta,\xi) = \dot W(t,x). \tag{2.12}$$

We readily obtain a *closed-form solution.*

$$u(t,x) = \frac{1}{2}\widetilde W\left(\frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}}\right) = \int_{C(t,x)} dW(s,y) = \left(\dot W, \mathbf{1}_{C(t,x)}\right)^1. \tag{2.13}$$

Here $C(t,x)$ is the domain of dependence of the point (t,x): $\{(s,y) : 0 \le s \le t, |y-x| \le t-s\}$.

Exercise 2.2. *Can we find a closed-form solution for the following equation?*

$$u_{tt} = u_{xx} + u\dot w(t), t > 0, x \in \mathbb{R}. \tag{2.14}$$

Here $w(t)$ is a standard Brownian motion.

2.3. **Linear equations with variable coefficients.** Consider the following stochastic wave equation in the normal triple (V, H, V') of Hilbert spaces, for $0 < t \le T$

$$\partial_t^2 u = \mathcal{A}(t)u + \mathcal{A}_1(t)u + \mathcal{B}(t)\partial_t u + f(t) + \sum_{k\ge 1}\left(\mathcal{M}_k(t)u + \mathcal{N}_k(t)\dot u + g_k(t)\right)\dot w_k(t), \tag{2.15}$$

$$u(0) = u_0, \quad \dot u(0) = v_0. \tag{2.16}$$

[1] Sometimes it is also written as $\mathcal{W}(C(t,x))$. The space-time white noise on $[0,T]\times\mathbb{R}$ defines an orthogonal random measure $\mathcal{W}$ on $\mathcal{B}([0,T]\times\mathbb{R})$ such that (i) for any $A \in \mathcal{B}([0,T]\times\mathbb{R})$, $\mathcal{W}(A) \sim \mathcal{N}(0,|A|)$, where $|A|$ is the volume of the set A and (ii)$\mathcal{W}(A+B) = \mathcal{W}(A) + \mathcal{W}(B)$ if $A, B \in \mathcal{B}([0,T]\times\mathbb{R})$ and $A \cap B = \emptyset$.

Denote $u = u_0 + \int_0^t v(s)\,ds$, which lies in $L^2(\Omega\times(0,T);H)$. Then $v \in L^2(\Omega\times(0,T));V')$ and

$$\begin{aligned} v(t) =& v_0 + \int_0^t \mathcal{A}u(s)ds + \int_0^t \mathcal{A}_1 u(s)ds + \int_0^t \mathcal{B}v(s)ds + \int_0^t f(s)ds \\ &+ \sum_{k\geq 1}\int_0^t (\mathcal{M}_k u(s) + \mathcal{N}_k v(s) + g_k(s))\,dw_k(s). \end{aligned} \tag{2.17}$$

Definition 2.3. *Let X, Y be separable Banach spaces and let $A = \{A(t),\, 0\leq t\leq T\}$ be a family of mappings from X to Y. The family is called*

- *(X,Y)-measurable if, for every $x\in X$ and $\ell\in Y'$ the real-valued function $t\to \ell(A(t)x)$ is measurable.*
- *(X,Y)-uniformly bounded if there exists a positive number C such that $\sup_{t\in[0,T],x\in X}\|A(t)x\|_Y \leq C\|x\|_X$.*

Definition 2.4. *Consider two separable Banach spaces X, Y and a family of mappings from X to Y $A = \{A(\omega,t),\,\omega\in\Omega,\,0\leq t\leq T\}$. The family is called*

- *(X,Y)-adapted if, for every $x\in X$ and $\ell\in Y'$ the real-valued function $t\to\ell(A(t)x)$ is $\mathcal{F}_t$ adapted.*
- *$L^\infty(X,Y)$-uniformly bounded if there exists a positive number C (non-random) such that $\sup_{x\in X,t\in[0,T],\omega\in\Omega}\|A(t)x\|_Y \leq C\|x\|_X$.*

Assume the following conditions on the operators and coefficients

- The family of operators $A = \{A(t)\}$, $t\in[0,T]$ is (V,V')-adapted and $L^\infty(V,V')$-uniformly bounded.
- The family of operators $\mathcal{A}_1 = \{\mathcal{A}_1(t)\}$, $t\in[0,T]$ and $\mathcal{M}_k = \{\mathcal{M}_k(t)\}$ are (V,H)-adapted and $L^\infty(V,H)$-uniformly bounded.
- The family of operators $\mathcal{B} = \{\mathcal{B}(t)\}$, $t\in[0,T]$ and $\mathcal{N}_k = \{\mathcal{N}_k(t)\}$ are (H,H)-adapted and $L^\infty(H,H)$-uniformly bounded.
- The following hold for every $v\in L^2(\Omega\times(0,T);V)$ and $h\in L^2(\Omega\times(0,T);H)$,

$$\sum_{k\geq 1}\mathbb{E}[\int_0^T \|\mathcal{M}_k v\|_H^2]\,dt \leq C\mathbb{E}[\int_0^T \|v(t)\|_V^2]\,dt, \tag{2.18}$$

$$\sum_{k\geq 1}\mathbb{E}[\int_0^T \|\mathcal{N}_k h\|_H^2]\,dt \leq C\mathbb{E}[\int_0^T \|h(t)\|_H^2]\,dt. \tag{2.19}$$

- $\mathcal{A} = \mathcal{A}^*$, i.e., $[\mathcal{A}u,v] = [u,\mathcal{A}v]$ for all $u,v\in V$ and $t\in[0,T]$ and all $\omega\in\Omega$.
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$$[\mathcal{A}u,u] + c_{\mathcal{A}}\|u\|_V^2 \leq M\|u\|_H^2 \tag{2.20}$$

- the family of operators A is differentiable for all $\omega\in\Omega$ and the derivative of A is (V,V')-adapted and $L^\infty(V;V')$-uniformly bounded.

Then for every $\mathcal{F}_0$-measurable initial conditions $u(0,x)\in L^2(\Omega;V)$ and $\dot{u}(0) = v_0\in L^2(\Omega;H)$ and $\mathcal{F}_t$-adapted free terms $f\in L^2(\Omega\times(0,T);H)$ and g_k satisfying $\sum_{k\geq 1}\|g_k\|^2_{L^2(\Omega\times(0,T);H)} < \infty$, Equation (2.17) has a solution, which is unique in $L_2(\Omega;\mathcal{C}(0,T);V)\times L_2(\Omega;\mathcal{C}(0,T);H)$. Morever, we may obtain

$$\begin{aligned} \|u(t)\|^2_{L_2(\Omega;\mathcal{C}(0,T);V))} + \|v(t)\|^2_{L_2(\Omega;\mathcal{C}(0,T);H))} \leq C(T)\Big(\|u_0\|^2_{L_2(\Omega;V)} \\ + \|v_0\|^2_{L_2(\Omega;H)} + \|f\|^2_{L_2(\Omega\times(0,T);H)} + \textstyle\sum_{k\geq 1}\|g_k\|^2_{L_2(\Omega\times(0,T);H)}\Big). \end{aligned} \tag{2.21}$$

2.4. **Nonlinear equations on $\mathbb{R}$.**

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} + f(x,t,u) + g(x,t,u)\dot{W}, \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = v_0(x); \quad x \in \mathbb{R} \end{cases} \tag{2.22}$$

where $c > 0$ is a constant and $\dot{W}$ is a standard white noise on $[0,T] \times \mathbb{R}$. Suppose that u_0 and v_0 are deterministic Hölder-continuous functions of order at least $\frac{1}{2}$.

Assume that f and g are locally Lipschitz continuous, i.e., for each N, there is a constant L_N such that for all $x, y, x', y' \in [-N, N]$ and all $z, z' \in \mathbb{R}$, and if $\xi = (x,y,z)$ and $\xi' = (x', y', z')$

$$\left|f(\xi) - f\left(\xi'\right)\right| + \left|g(\xi) - g\left(\xi'\right)\right| \le L_N \left|\xi - \xi'\right|, \quad |f(\xi)| + |g(\xi)| \le L_N(1 + |\xi|). \tag{2.23}$$

The mild solution of the problem is written as

$$u(x,t) = \frac{1}{2}\Big(u_0(x+ct) + u_0(x-ct)\Big) + \frac{1}{2c}\Big(\int_{x-ct}^{x+ct} v_0(y)dy\Big) \tag{2.24}$$

$$+\frac{1}{2c}\Big(\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)dsd\tau\Big) + \frac{1}{2c}\Big(\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} g(y,s,u(y,s))\mathcal{W}(dyds)\Big). \tag{2.25}$$

Theorem 2.5 (Hölder continuity)**.** *Let u be the solution of* (2.22)*. Then for any N and $p \ge 1$, there is a constant $K_{N,p}$ such that: $\mathbb{E}[|u(t,x)|^{2p}] \le K_{N,p}$ for $(t,x) \in C(N,0)$ and $\mathbb{E}[|u(t,x) - u(s,y)|^{2p}] \le K_{N,p}(|x-y|^p + |t-s|^p)$ for (t,x), $(s,y) \in C(N,0)$. Here $C(N,0)$ is defined in* (2.13)*.*

2.4.1. *Derivation of a finite difference scheme.* Assume $c = 1$. Applying the characteristic transform (2.11), we have

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 2\frac{\partial^2 u}{\partial \xi \partial \eta}. \tag{2.26}$$

The equation (2.22) becomes

$$2\frac{\partial^2 u}{\partial \xi \partial \eta} = f + g\dot{W}. \tag{2.27}$$

Let $\Diamond$ be the diamond with corners at $(x, t+h)$, $(x+h, t)$, $(x-h, t)$, and $(x, t-h)$. Integrating both sides of (2.27) over $\Diamond$ leads to the following

$$\begin{aligned} 2[u(x,t+h) - u(x+h,t) - u(x-h,t) + u(x,t-h)] &= 2\int_{\Diamond} \frac{\partial^2 u}{\partial \xi \partial \eta} d\xi d\eta \\ &= \int_{\Diamond} f(y,s,u)dyds + \int_{\Diamond} g(y,s,u)\mathcal{W}(dyds). \end{aligned}$$

Denote the area of $\Diamond$ by $|\Diamond| = 2h^2$.

$$\begin{aligned} \int_{\Diamond} f(y,s,u)dyds &\approx f\left(x, t, \frac{1}{2}(u(x+h,t) + u(x-h,t))\right)|\Diamond|, \\ \int_{\Diamond} g(y,s,u)W(dyds) &\approx g\left(x, t, \frac{1}{2}(u(x+h,t) + u(x-h,t))\right)\mathcal{W}(\Diamond). \end{aligned}$$

Here $\mathcal{W}(\Diamond) = \int_{\Diamond} \mathcal{W}(dyds)$ and $\mathcal{W}$ is an orthogonal measure on $\mathcal{B}([0,T] \times \mathbb{R})$ such that $\mathcal{W}(dyds) \sim \mathcal{N}(0, dyds)$ and $\mathcal{W}(A+B) = \mathcal{W}(A) + \mathcal{W}(B)$ if $A, B \in \mathcal{B}([0,T] \times \mathbb{R})$ and $A \cap B = \emptyset$.

Then we arrive that the following 'scheme'

$$\begin{aligned} u(x,t+h) &\approx u(x+h,t)+u(x-h,t)-u(x,t-h)+\frac{1}{2}f\left(x,t,\frac{1}{2}(u(x+h,t)+u(x-h,t))\right)|\Diamond| \\ &+\frac{1}{2}g\left(x,t,\frac{1}{2}(u(x+h,t)+u(x-h,t))\right)\mathcal{W}(\Diamond). \end{aligned} \tag{2.28}$$

Let $h>0$, put $x_i=ih$, $t_j=jh$, and define subsets L_h and M_h of $h\mathbb{Z}^2$ by

$$\mathcal{L}_h=\{(x_i,t_j): i,j\in\mathbb{Z}, ij \text{ is even }\},\quad \mathcal{M}_h=\{(x_i,t_j): i,j\in\mathbb{Z}, ij \text{ is odd}\}. \tag{2.29}$$

The finite difference scheme is then

$$\begin{aligned} u_{i,j+1} &= u_{i+1,j}+u_{i-1,j}-u_{i,j-1}+h^2 f\left(x_i,t_j,\frac{1}{2}\left(u_{i+1,j}+u_{i-1,j}\right)\right) \\ &+\frac{1}{2}g\left(x_i,t_j,\frac{1}{2}\left(u_{i+1,j}+u_{i-1,j}\right)\right)\mathcal{W}\left(\Diamond_{ij}\right). \end{aligned} \tag{2.30}$$

As we need the values at $u_{i,-1}$, we may use the D'Alembert solution to construct a 'ghost' point as follows.

$$u_{i,-1}\equiv u\left(x_i,t_{-1}\right)=\frac{1}{2}\left(u_0\left(x_{i-1}\right)+u_0\left(x_{i+1}\right)\right)-\frac{1}{2}\int_{x_{i-1}}^{x_{i+1}} v_0(y)dy, \tag{2.31}$$

which is from

$$u(x,t)=\frac{1}{2}\left(u_0(x-t)+u_0(x+t)\right)+\frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy,\quad t<0, x\in\mathbb{R}. \tag{2.32}$$

Here we may also use $u_{i,-1}=\frac{u_0(x_{i+1})+u_0(x_{i-1})}{2}-hv_0(x_i)$. Then we summarize the scheme

$$\begin{aligned} u_{i,-1} &= \frac{1}{2}\left(u_0\left(x_{i-1}\right)+u_0\left(x_{i+1}\right)\right)-\frac{1}{2}\int_{x_{i-1}}^{x_{i+1}} v_0(y)dy,\quad i \text{ odd}, \\ u_{i,1} &= u_0\left(x_{i-1}\right)+u_0\left(x_{i+1}\right)-u_{i,-1}+\frac{h^2}{2}f\left(x_i,0,\frac{1}{2}\left(u_0\left(x_{i+1}\right)\right)W\left(\Diamond_{i0}-1\right)\right), \\ &+\frac{1}{2}g\left(x_i,0,\frac{1}{2}\left(u_0\left(x_{i+1}\right)+u_0\left(x_{i-1}\right)\right)\right)\mathcal{W}\left(\Diamond_{i0}\cap\mathbb{R}\times\mathbb{R}_+\right),\quad i \text{ odd}, \\ u_{i,j+1} &= 2u_{i+1,j}+u_{i-1,j}-u_{i,j-1}+h^2 f\left(x_i,t_j,\frac{1}{2}\left(u_{i+1,j}+u_{i-1,j}\right)\right) \\ &+ \frac{1}{2}g\left(x_i,t_j,\frac{1}{2}\left(u_{i+1,j}+u_{i-1,j}\right)\right)\mathcal{W}\left(\Diamond_{ij}\right),\quad (x_i,t_{j+1})\in\mathcal{L}_h,\quad j\geq 1. \end{aligned} \tag{2.33}$$

Theorem 2.6 (Convergence order)**.** *Let u_{ij} be the solution of the finite difference scheme for $h>0$. Then (i) $u_{i,j}$ converges as $h\to 0$ in all L^p to the true solution, and, for $p\geq 1$ there exists $K_{N,p}$ such that*

$$\mathbb{E}[|u_{i,j}-u(x_i,t_j)|^p]\leq K_p h^{p/2},\quad (x_i,t_j)\in\mathcal{L}_h\cap C(N,0). \tag{2.34}$$

Theorem 2.7 (lower bound)**.** *The half-order is best possible on a uniform rectangular lattice in the following sense. Let h and δt be strictly positive, and let $\Box_{ij}=(ih,(i+1)h]\times(j\delta t,(j+1)\delta t]$. Then there are Lipschitz functions f and g such that any numerical scheme for (2.22) which depends only on the increments $W(\Box_{ij})$ cannot converge in L^2 at a rate faster than half-order. The same is true if $\Box_{ij}$ is replaced by $\Diamond_{ij}$.*

Remark 2.8. *It is possible to achieve higher-order convergence if the equation is diagonalizable, i.e., the leading-order operator $-u_{xx}$ and the covariance operator Q (if $W^Q(t,x)$ is considered) have only point spectrum and the same set of eigenfunctions, See Lecture 9 for the exponential Euler scheme. When $Q = I$, we can always take the form $\dot W(t,x) = \sum_{k=1}^{\infty} e_k(x)\dot W_k(t)$, where $e_k(x)$'s are the eigenfunctions of u_{xx}.*

3. Two-dimensional equations

Here we consider the *two-dimensional* ($d = 2$) passive scalar equation with periodic boundary conditions:

$$(3.1)\quad \begin{aligned} du(t,x) + \sum_{k=1}^{\infty}\sum_{i=1}^{d} \sigma_k^i(x) D_i u \circ dW_k(t) &= 0,\ t>0,\ x\in(0,\ell)^2,\\ u(t,x^1+\ell,x^2) &= u(t,x^1,x^2+\ell) = u(t,x),\ t>0,\ x\in(0,\ell)^2,\\ u(0,x) &= u_0(x),\quad x\in(0,\ell)^2,\end{aligned}$$

where '$\circ$' indicates the Stratonovich product, $\ell > 0$, the initial condition $u_0(x)$ is a periodic function with the period $(0,\ell)^2$, and $\sigma_k^i(x)$ are divergence-free periodic functions with the period $(0,\ell)^2$:

$$(3.2)\qquad \operatorname{div}\sigma_k = 0.$$

In (3.1), we take a combination of such $\sigma_k(x)$ so that the corresponding spatial covariance C is symmetric and stationary: $C(x-y) = \sum_{k=1}^{\infty} \lambda_k \sigma_k(x)\sigma_k^\top(y)$, where λ_k are some non-negative numbers. We rewrite (3.1) in the Ito's form:

$$(3.3)\quad \begin{aligned} du(t) + \frac{1}{2}\sum_{i,j=1}^{d} C_{ij}(0) D_i D_j u dt &+ \sum_{k=1}^{\infty}\sum_{i=1}^{d}\sigma_k^i(x) D_i u dW_k(t) = 0,\\ u(t,x^1+\ell,x^2) &= u(t,x^1,x^2+\ell) = u(t,x),\ t>0,\ x\in(0,\ell)^2,\\ u(0,x) &= u_0(x),\quad x\in(0,\ell)^2.\end{aligned}$$

The solution $u(t,x)$ of (3.1) has the following (conditional) probabilistic representation

$$(3.4)\qquad u(t,x) = u_0(X_{t,x}(0)),\ \text{a.s.},$$

where $X_{t,x}(s)$, $0 \le s \le t$, is the solution of the system of (backward) characteristics

$$(3.5)\qquad -dX = \sum_k \sigma_k(X)\overleftarrow{dW_k}(s),\quad X(t) = x.$$

Due to (3.2), it holds that

$$(3.6)\qquad \sum_k \frac{\partial \sigma_k}{\partial x}\sigma_k = 0,$$

the phase flow of (3.5) preserves phase volume (see e.g. [Milstein and Tretyakov, 2004, p. 247, Equation (5.5)]). We also recall that the Ito and Stratonovich forms of (3.5) coincide. As shown in [Milstein and Tretyakov, 2004], it is beneficial to approximate (3.5) using phase volume preserving schemes [Milstein and Tretyakov, 2004, Chapter 4], e.g., by the midpoint method. The midpoint method for (3.5) takes the form (here the Ito and Stratonovich forms of (3.5) coincide): for an integer $m \ge 1$,

$$(3.7)\qquad X_m = x,$$

$$X_l = X_{l+1} + \sum_k \sigma_k \left(\frac{X_l + X_{l+1}}{2}\right) \left(\zeta_k^{\Delta t}\right)_l \sqrt{\Delta t},\ l = n-1,\dots,0,$$

where $\left(\zeta_k^{\Delta t}\right)_l$ are, e.g., i.i.d. random variables with the law

$$\zeta_k^{\Delta t} = \begin{cases} \xi_k,\ |\xi_k| \le A_{\Delta t}, \\ A_{\Delta t},\ \xi_k > A_{\Delta t}, \\ -A_{\Delta t},\ \xi_k < -A_{\Delta t}, \end{cases} \tag{3.8}$$

and ξ_k are i.i.d standard Gaussian random variables, and $A_{\Delta t} = \sqrt{2c|\ln \Delta t|}$, $c \ge 1$. Its weak order is equal to one [Milstein and Tretyakov, 2004].

For numerical results using WCE and methods of probabilistic characteristics for the passive scalar equation, see Chapter 5 of [Zhang and Karniadakis, 2017].

Appendix A. Proof of Theorem 1.3

Proof. (of Theorem 1.3) We first prove the strong convergence. Note that

$$\mathbb{E}[(u-u_{\mathsf{n}})^2] = (u_0(x-t)-1)^2\mathbb{E}[(\exp(\sigma W(x)-\sigma W(x-t))-\exp(\sigma W_{\mathsf{n}}(x)-\sigma W_{\mathsf{n}}(x-t)))^2]. \tag{A.1}$$

By the fact that $\exp(a) - \exp(b) = \exp(\theta a + (1-\theta)b)(a-b)$ where $a \le \theta \le b$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \mathbb{E}[(u-u_{\mathsf{n}})^2] \\ = \ & (u_0(x-t)-1)^2\mathbb{E}[\big(\exp(\sigma W(x)-\sigma W(x-t))-\exp(\sigma W_{\mathsf{n}}(x)-\sigma W_{\mathsf{n}}(x-t))\big)^2] \\ \le \ & (u_0(x-t)-1)^2\big(\mathbb{E}[(\exp(4\sigma\theta(W(x)-W(x-t))+4\sigma(1-\theta)(W_{\mathsf{n}}(x)-W_{\mathsf{n}}(x-t))]\big)^{1/2} \\ & \times\sigma^2\big(\mathbb{E}[((W(x)-W(x-t))-(W_{\mathsf{n}}(x)-W_{\mathsf{n}}(x-t)))^4]\big)^{1/2}. \end{aligned}$$

It requires to estimate the two expectations in this inequality. The first one is bounded as follows:

$$\begin{aligned} & (\mathbb{E}[\exp(4\sigma\theta(W(x)-W(x-t))+4(1-\theta)\sigma(W_{\mathsf{n}}(x)-W_{\mathsf{n}}(x-t))])^{1/2} \\ \le \ & (\mathbb{E}[\exp(8\sigma\theta(W(x)-W(x-t)))])^{1/4}(\mathbb{E}[\exp(8(1-\theta)\sigma(W_{\mathsf{n}}(x)-W_{\mathsf{n}}(x-t)))])^{1/4} \\ \le \ & \exp(8\sigma^2\theta^2 t)\exp(8(1-\theta)^2\sigma^2 t) \le \exp(8\sigma^2 t). \end{aligned} \tag{A.2}$$

Now we estimate the second expectation $\mathbb{E}[(W(x)-W(x-t)-(W_{\mathsf{n}}(x)-W_{\mathsf{n}}(x-t)))^4]$. In fact,

$$\begin{aligned} & \mathbb{E}[(W(x)-W(x-t)-(W_{\mathsf{n}}(x)-W_{\mathsf{n}}(x-t)))^4] \\ = \ & \mathbb{E}[(\sum_{k=\mathsf{n}+1}^{\infty}[M_k(x)-M_k(x-t)]\xi_k)^4] \\ = \ & \mathbb{E}[(\sum_{k=\mathsf{n}+1}^{\infty}\sum_{l=\mathsf{n}+1}^{\infty}[M_k(x)-M_k(x-t)]^2[M_l(x)-M_l(x-t)]^2\xi_k^2\xi_l^2] \\ \le \ & 3\sum_{k=\mathsf{n}+1}^{\infty}\sum_{l=\mathsf{n}+1}^{\infty}[M_k(x)-M_k(x-t)]^2[M_l(x)-M_l(x-t)]^2 \\ = \ & 3(\sum_{k=\mathsf{n}+1}^{\infty}[M_k(x)-M_k(x-t)]^2)^2 \le C\frac{1}{\mathsf{n}^2}, \end{aligned} \tag{A.3}$$

where $M_k = \int_0^x m_k(y)\,dy$ with $m_1(x) = \frac{1}{\sqrt{L}}$, $m_k(x) = \sqrt{\frac{2}{L}}\cos(\frac{\pi(k-1)x}{L})$ and C depends only on t and x.

By (A.2) and (A.3), we have the first estimate in (1.6).

Now we prove the weak convergence. It suffices to check $\mathbb{E}[(u-1)^k] - \mathbb{E}[(u_{\mathsf{n}}-1)^k]$. By (1.2) and (1.5), we have

$$\begin{aligned}\left|\mathbb{E}[(u-1)^k] - \mathbb{E}[(u_{\mathsf{n}}-1)^k]\right| &= \Big|(u_0(x-t)-1)^k \exp(\frac{k^2}{2}\sigma^2\mathbb{E}[(W(x)-W(x-t))^2]) \\ &\quad -(u_0(x-t)-1)^k \exp(\frac{k^2}{2}\sigma^2\mathbb{E}[(W_{\mathsf{n}}(x)-W_{\mathsf{n}}(x-t))^2])\Big| \\ &\le |u_0(x-t)-1|^k \exp(\frac{k^2}{2}\sigma^2\mathbb{E}[(W(x)-W(x-t))^2]) \\ &\quad \times\frac{k^2}{2}\sigma^2\big(\mathbb{E}[(W(x)-W(x-t))^2] - \mathbb{E}[(W_{\mathsf{n}}(x)-W_{\mathsf{n}}(x-t))^2]\big),\end{aligned}$$

where we have used the fact $e^x - e^y = e^{\theta x+(1-\theta)y}(x-y)$ $(0\le\theta\le 1)$ and that $\mathbb{E}[(W_{\mathsf{n}}(x) - W_{\mathsf{n}}(x-t))^2] \le \mathbb{E}[(W(x)-W(x-t))^2]$. By $\mathbb{E}[(W(x)-W(x-t))^2]-\mathbb{E}[(W_{\mathsf{n}}(x)-W_{\mathsf{n}}(x-t))^2] = \sum_{k=\mathsf{n}+1}^{\infty}\big(M_k(x)-M_k(x-t)\big)^2$, we then have

$$\left|\mathbb{E}[(u-1)^k] - \mathbb{E}[(u_{\mathsf{n}}-1)^k]\right| \le \frac{k^2}{2}\sigma^2\left|(u_0(x-t)-1)^k\right|\exp(\frac{k^2}{2}\sigma^2 t)\frac{C}{\mathsf{n}}.$$

Hence, the estimate of the weak convergence order follows. □

The upwind scheme is only of first-order convergence even when $\sigma = 0$. For higher-order schemes, we refer to Chapter 6 of [Gustafsson et al., 1995].

Appendix B. Derivation of the modified equation (1.7)

First, perform the Taylor's expansion about (x_j, t_n).

$$\begin{aligned} u_j^{n+1} &= u_j^n + \delta t(\partial_t u)_j^n + \frac{(\delta t)^2}{2}(\partial_t^2 u)_j^n + \frac{(\delta t)^3}{6}(\partial_t^3 u)_j^n + \cdots \\ u_{i-1}^n &= u_j^n - \delta x(\partial_x u)_j^n + \frac{(\delta x)^2}{2}(\partial_x^2 u)_j^n - \frac{(\delta x)^3}{6}(\partial_x^3 u)_j^n + \cdots\end{aligned}$$

Then substitution into the difference equation (scheme) gives

$$(\partial_t u)_j^n + c(\partial_x u)_j^n = \frac{\sigma^2\delta t}{2}(u_j^n - 1) + \sigma(u_j^n-1)\Delta W_n \tag{B.1}$$

$$-\frac{\delta t}{2}(\partial_t^2 u)_j^n - \frac{(\delta t)^2}{6}(\partial_t^3 u)_j^n - c\frac{\delta x}{2}(\partial_x^2 u)_j^n - c\frac{(\delta x)^2}{6}(\partial_x^3 u)_j^n\cdots \tag{B.2}$$

From here, we can tell the convergence order in δt and δx (but the modified equation requires even more).

Second, let's derive the modified equation when $\sigma = 0$. We have obtained

$$\partial_t u + c\partial_x u = -\frac{\delta t}{2}\partial_t^2 u - \frac{(\delta t)^2}{6}\partial_t^3 u + c\frac{\delta x}{2}\partial_x^2 u - c\frac{(\delta x)^2}{6}\partial_x^3 u + \cdots \tag{B.3}$$

Using the facts (which can derived from the equation above by differentiating it in t, x and t, x),

$$\partial_t^3 u = -c^3\partial_x^3 u + O(\delta t, \delta x), \tag{B.4}$$

$$\partial_t^2 u = c^2\partial_x^2 u + c^2\delta x(\rho-1)\partial_x^3 u + O(\delta t, \delta x), \quad \rho = c\frac{\delta t}{\delta x}, \tag{B.5}$$

we then obtain the modified equation (1.7).

Appendix C. Characteristic decomposition

Consider the following system of equations

$$\partial_t v = A\partial_x v, \quad x \in (0,1).$$

Let $A = S\Lambda S^{-1}$ where $\Lambda = \begin{pmatrix} \Lambda^+ & 0 & 0 \\ 0 & \Lambda^- & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we can let $u = S^{-1}v$ and obtain that

$$\partial_t u = \Lambda \partial_x u.$$

where we assume that A is independent of x and t. We then have separable/uncoupled equations to solve. Then we can apply numerical methods for scalar equations. See Chapter 6 of [Gustafsson et al., 1995] for more discussions.

References

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