Capacitance Matrix Method

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Thanks to **Maksymilian Dryja**, University of Warsaw

Outline

1. Timeline
2. Capacitance Matrix Method-CMM
3. Applications
4. Domain Imbedding
5. Proskurowski-Widlund 76'
6. Domain Decomposition-Iterative Substructuring
7. Conclusion

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Timeline 49-86's

• In 1949, Sherman-Morrison-**Woodbury identity**

$$(A + UDV^T)^{-1} = A^{-1} - [A^{-1}U(D^{-1} + V^T A^{-1}U)^{-1}V^T A^{-1}]$$

• In 1952, **conjugated gradient method** introduced by Hestenes-Stiefel

• In the 1960th years, there existed **fast solvers** for FD discretizations of some elliptic problems on rectangular regions based on **FFT**

• In 1968, the description of the **CMM** is credited to Oscar Buneman (see R. W. Hockney *Formation and stability of virtual electrodes in a cylinder*)

Timeline 70-84's

- Early 70's: Buzbee, Dorr, George, Golub, Hockney, others
- In 1976, Proskurowski-Widlund, *On the numerical solution of Helmholtz's equation by CMM*
- Domain Imbedding: Shieh 1978 and 1979, O'Leary-Widlund 1979, Proskurowski-Widlund 1980 (FEM), others
- Pierre-Louis Lions, 1979. Variational Alternating Schwarz
- Ficticius Component Method: Astrakhantsev 1978 and 1985, Matsokin and Skripko 1983, Matsokin and Nepomnyaschikh 1985
- Domain Decomposition: Bjørstad-Widlund 1981 and 1986, Dryja 1982 and 1984, Dihn-Glowinski-Périaux 1985, others
- In 1982, Dryja, *A capacitance matrix method for Dirichlet problem on polygon region*

Timeline 85-87's

- In 1985, W. Hackbusch, *Multi-Grid Methods and Applications*
- In 1986, H. Yserentant, *On the multilevel splitting of finite element spaces*.
- In 1986, Bramble-Basciak-Schatz, *The construction of precond. for elliptic problems by substructures I*
- In January 1987, Paris, First International Symposium of Domain Decomposition Methods.

Capacitance Matrix Method-CMM

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CMM: B is a Fast Solver

- Let $A \in \mathbb{R}^{n\times n}$ invertible matrix. Find the solution x of

$$Ax = b$$

- Let $B \in \mathbb{R}^{n\times n}$ invertible matrix which has the same rows as the matrix A, except the last p rows (in general $p \ll n$)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad B = \begin{bmatrix} A_{11} & A_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

- See that

$$Bx = \begin{bmatrix} b_1 \\ * \end{bmatrix}$$

CMM: Capacitance Matrix

- We look for solution of the form

$$x = B^{-1}\hat{b} + B^{-1}I_{np}\,w_p \qquad \text{where}$$

$$\hat{b} = \begin{bmatrix} b_1 \\ \hat{b}_2 \end{bmatrix} \qquad \hat{b}_2 - \text{arbitrary} \qquad I_{np} = \begin{bmatrix} 0 \\ I_{pp} \end{bmatrix}$$

- After some algebra, w_p satisfies $Cw_p = g_p$ where

$$C = I_{pp} - I_{np}^T(B-A)B^{-1}I_{np} \text{ and } g_p = b_2 - \hat{b}_2 + I_{np}^T(B-A)B^{-1}\hat{b}$$

or $$C = I_{np}^T AB^{-1}I_{np} \text{ and } g_p = b_2 - I_{np}^T AB^{-1}\hat{b}$$

- $C \in \mathbb{R}^{p\times p}$ is invertible and called the **Capacitance Matrix**

Implementations

- To find x, solve $Cw_p = g_p$ and $x = B^{-1}\hat{b} + B^{-1}I_{np}w_p$
- To solve $Cw_p = g_p$ use a iterative method

  Each vector multiplication to C requires a B^{-1} fast solver
- Can build C by multiplying $Ce_i, 1 \leq i \leq p$

  To build C: cost $O(p\,n\log_2(n)$

  Factorization of C: cost $O(p^3)$
- With an optimal preconditioner $\kappa(P^{-1}C) = O(1)$

  Cost $O(n)\log_2(n)$

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Applications

- Capacitance Matrix Method:

  Domain Imbedding-DI: After 1969. $B \rightarrow$ FFT

  Domain Decomposition-Iterative Substructuring: After 1980. $B \rightarrow$ localization

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Domain Imbedding-Dirichlet

- Problem: $-\Delta u = f$ on Ω and $u = 0$ on $\partial\Omega$
- Rectangular region $R \supset \Omega$ and

$$\Omega_a \equiv \Omega \quad \Omega_b = R\backslash\overline{\Omega}_a \quad \overline{\Gamma} = \overline{\Omega}_a \cap \overline{\Omega}_b$$

- Finite Difference $(-\Delta_h x_a = b_a$ in $\Omega_a^h)$) and $(x_2 = 0$ on $\Gamma_h)$

$$Ax = \begin{bmatrix} B_{aa} & 0 & B_{a2} \\ 0 & B_{bb} & B_{b2} \\ 0 & 0 & I_{22} \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_2 \end{bmatrix} = \begin{bmatrix} b_a \\ 0 \\ 0 \end{bmatrix} = b$$

- FD $(-\Delta_h x = \hat{b}$ in $R_h)$ and (zero Dirichlet on ∂R)

$$B = \begin{bmatrix} B_{aa} & 0 & B_{a2} \\ 0 & B_{bb} & B_{b2} \\ B_{2a} & B_{2b} & B_{22} \end{bmatrix} \qquad \begin{matrix} \to \Omega_a^h \\ \to \Omega_b^h \\ \to \Gamma_h \end{matrix}$$

Domain Imbedding-Dirichlet

- The capacitance matrix

$$C = S^{-1} = (B_{22} - \sum_{i=\{a,b\}} B_{2i} B_{ii}^{-1} B_{i2})^{-1}$$

$$C = I_{pp} - I_{np}^T (B - A) B^{-1} I_{np} = I_{np}^T A B^{-1} I_{np}$$

- Preconditioners

$\kappa((-\Delta_p)^{\frac{1}{2}} C) = O(1)$. Dryja 1982

$\kappa(\mathcal{H}^{-1} S^{-1} \mathcal{H}^{-T}) = O(1 + \log^2 h)$. Yserentant 1986

Domain Imbedding-Neumann

- $-\Delta u + cu = f$ in Ω and $\partial_n u = 0$ on $\partial\Omega$

$$Ax = \begin{bmatrix} B_{aa} & 0 & B_{a2} \\ 0 & B_{bb} & B_{b2} \\ B_{2a} & 0 & B_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_2 \end{bmatrix} = \begin{bmatrix} b_a \\ 0 \\ 0 \end{bmatrix} = b$$

- Well conditioned Capacitance Matrix

$$C = S_a(S)^{-1} = (B_{22}^{(a)} - B_{2a}B_{aa}^{-1}B_{a2})(S)^{-1}$$

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Proskurowski-Widlund, 1976

- Modification of a rectangular domain

- R. W. Hocknew Five-points discretization of Laplace equation with a number of electrodes nodes are introduced in the interior of a rectangular region or on straight line segment to which one of several mesh points are assigned

- In Proskurowski-Widlund 76', the goal is to solve the Poisson problem in complex geometry (Dirichlet or Neumann problems) using FD.

- In Proskurowski-Widlund 76', the CMM is interpreted using classical potential theory using single-layer dipole (ansatz). And in this case C is SPD with condition number $O(h)$ and CG is used

Proskurowski-Widlund 76'

$$x = B^{-1}\hat{b} + B^{-1} I_{np}\, w_p$$

- This equation is interpreted using the 76' paper notation

$$u = Gf + GD\mu \qquad G \;\; \text{Discrete Green's Function}$$

Here μ is the dipole density (jump of normal derivatives)

$$C\mu = (I_p + Z^T GD)\mu = -Z^T Gf = g$$

- G is translated invariant (FFT techniques). To build C

$$O(nlog_2 n + p^2)$$

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L-Shaped Domain

- Poisson Equation on L-shaped region $\Omega = \Omega_a \cup \Gamma \cup \Omega_b$

$\Omega_a = (0, x_1)\times(0, y_2) \quad \Gamma = \{x_1\}\times(0, y_1) \quad \Omega_b = (x_1, x_2)\times(0, y_1)$

- FDM with Homogeneous Dirichlet BC. Solve $Ax = b$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \begin{matrix} \to \Omega_a^h \cup \Omega_b^h \\ \to \Gamma_h \end{matrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- We choose B of the form

$$B = \begin{bmatrix} A_{11} & A_{12} \\ 0 & I_{22} \end{bmatrix}$$

- $C = A_{22} - A_{21}A_{11}^{-1}A_{12} = S$, the Schur complement of A with respect to A_{22}

Remarks

- $\kappa(C) = O(1/h)$
- $\kappa((-\Delta_p)^{-\frac{1}{2}}C) = O(1)$. Dryja 1982
- If we choose

$$B = \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix}$$

$C = I_{22}$

Sort of BDDC with Vertex Constraint

- Several substructures Ω_i and interface Γ

$$\overline{\Omega} = \cup_{i=1}^{N} \overline{\Omega}_i \quad \Gamma = \Omega \backslash (\cup_{i=1}^{N} \Omega_i)$$

- Let V be the set of vertices of the substructures

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \begin{array}{r} \to (\Omega_h \backslash \Gamma_h) \cup V \\ \to \Gamma_h \backslash V \end{array} \qquad B = \begin{bmatrix} A_{11} & A_{12} \\ 0 & I_{22} \end{bmatrix}$$

- $C = A_{22} - A_{21} A_{11}^{-1} A_{12}$
- Preconditioner $K = \text{blockdiag}\{(-\Delta_{ij})^{\frac{1}{2}}\}_{F_{ij} \subset \Gamma}$

Neumann-Dirichlet Method

- Dryja, Proskurowski, Widlund, 1986.
- Checkerboard distribution local solvers. Or two subdomains

$$\overline{\Omega} = \overline{\Omega}^{(D)} \bigcup \overline{\Omega}^{(N)} = (\cup_{i \in N_D} \overline{\Omega}_i^{(D)}) \bigcup (\cup_{i \in N_N} \overline{\Omega}_i^{(N)})$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \begin{array}{l} \to \Omega_h^{(D)} \text{ (interior)} \\ \to \overline{\Omega}_h^{(N)} \end{array} \quad B = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22}^{(N)} \end{bmatrix}$$

$$v_2^T A_{22} u_2 = a_{\Omega^{(N)}}(u_2, v_2) + a_{\Omega^{(D)}}(u_2, v_2)$$

- $C = (A_{22} - A_{21} A_{11}^{-1} A_{12})(A_{22}^{(N)})^{-1}$
- $\kappa(S(A_{22}^{(N)})^{-1}) = O(1 + \log H/h)^2$

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Conclusions

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