Transformation

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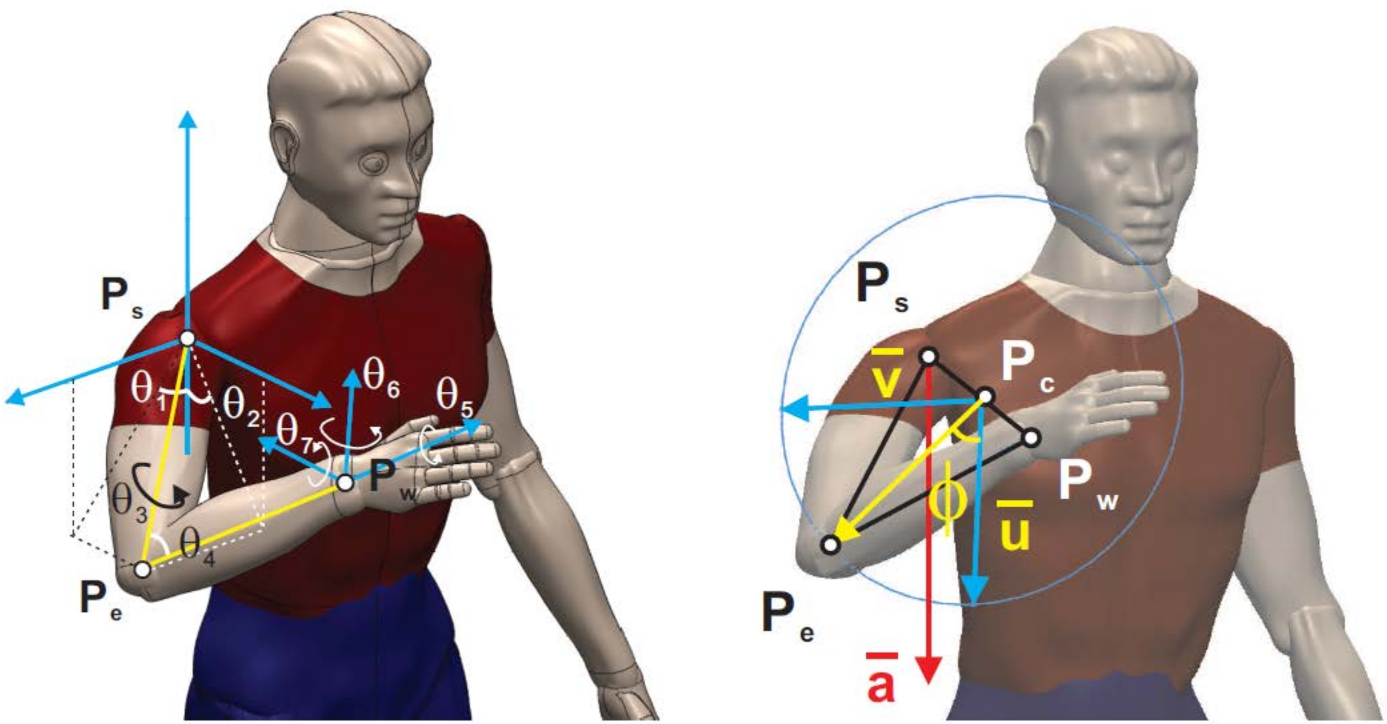
Announcement

- Project Presentation
  - # individual project VS # team project
  - All or some?

Representation of Rigid-body Configurations

- Parameterization matter!



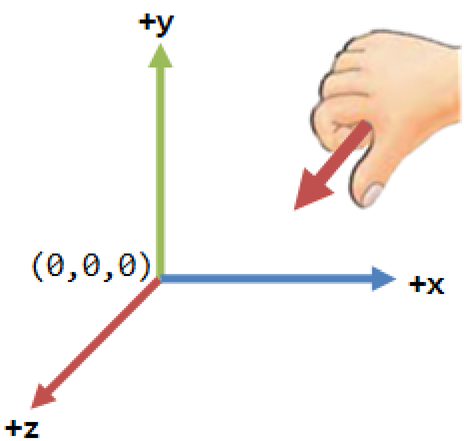
Representation of Rigid-body Configurations

- Many representations
  - Euler angle
  - Homogeneous transformation
  - Quaternions
- Different representations have pros and cons

Transformations

- An understanding of 2D and 3D rigid-body transformations is key to motion planning
- There is no "best" representation
  - Each representation is useful in a different way

Right-handed Coordinate System



Right-Hand System (RHS) or
Counter-Clockwise (CCW) System

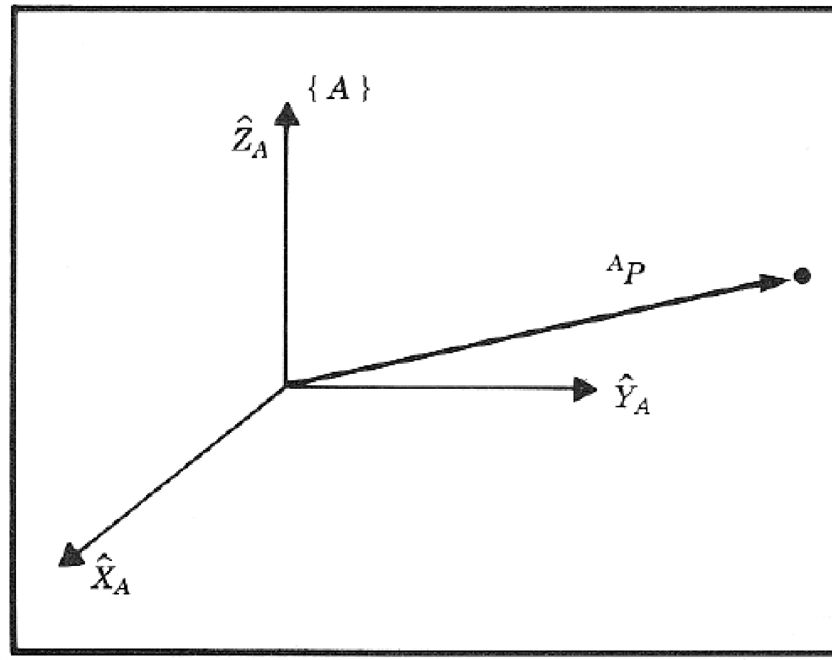
Representation for a Position

- 3X1 position vector in a reference coordinate system

Coordinate System

$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Position vector



Representation for a Rotation

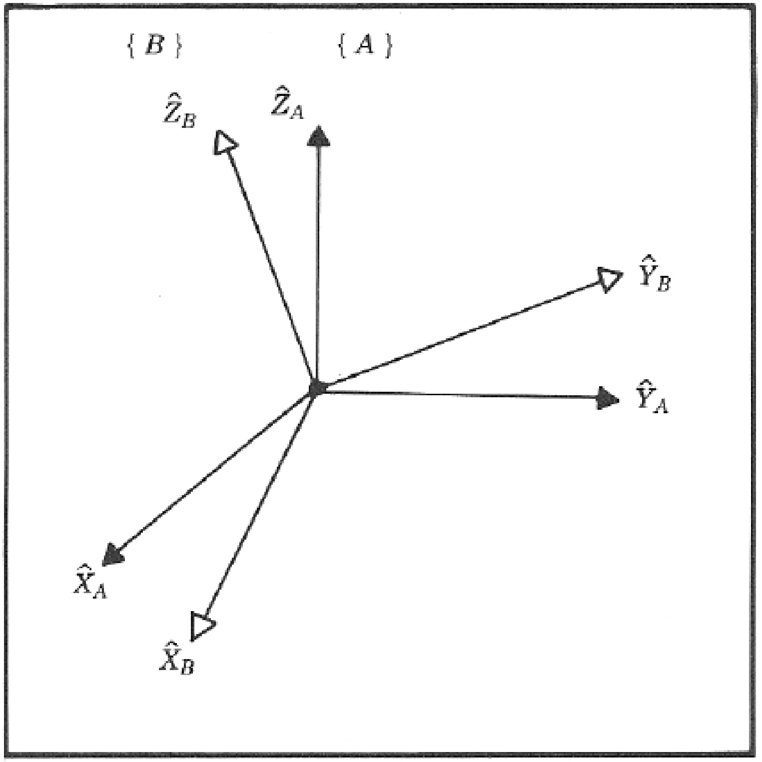
- Attach a frame to the Body **B**
- Rotation of Body **B** with respect to Frame **A**

The rotation matrix describing frame {B} relative to frame {A}

Reference Frame

$$ {}^{A}_{B}R = [{}^{A}\hat{X}_{B}, {}^{A}\hat{Y}_{B}, {}^{A}\hat{Z}_{B}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

Body Frame



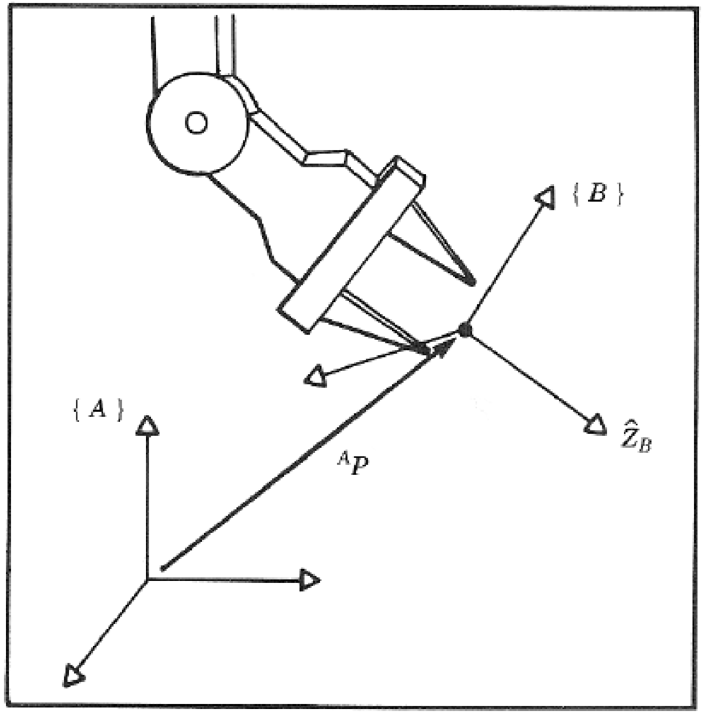
Representation of a Frame

- Complete specification of a 3D object
  - Position + Orientation

$$\{B\} = \{{}^{A}_{B}R, {}^{A}P_{BORG}\}$$

The rotation matrix describing frame {B} relative to frame {A}

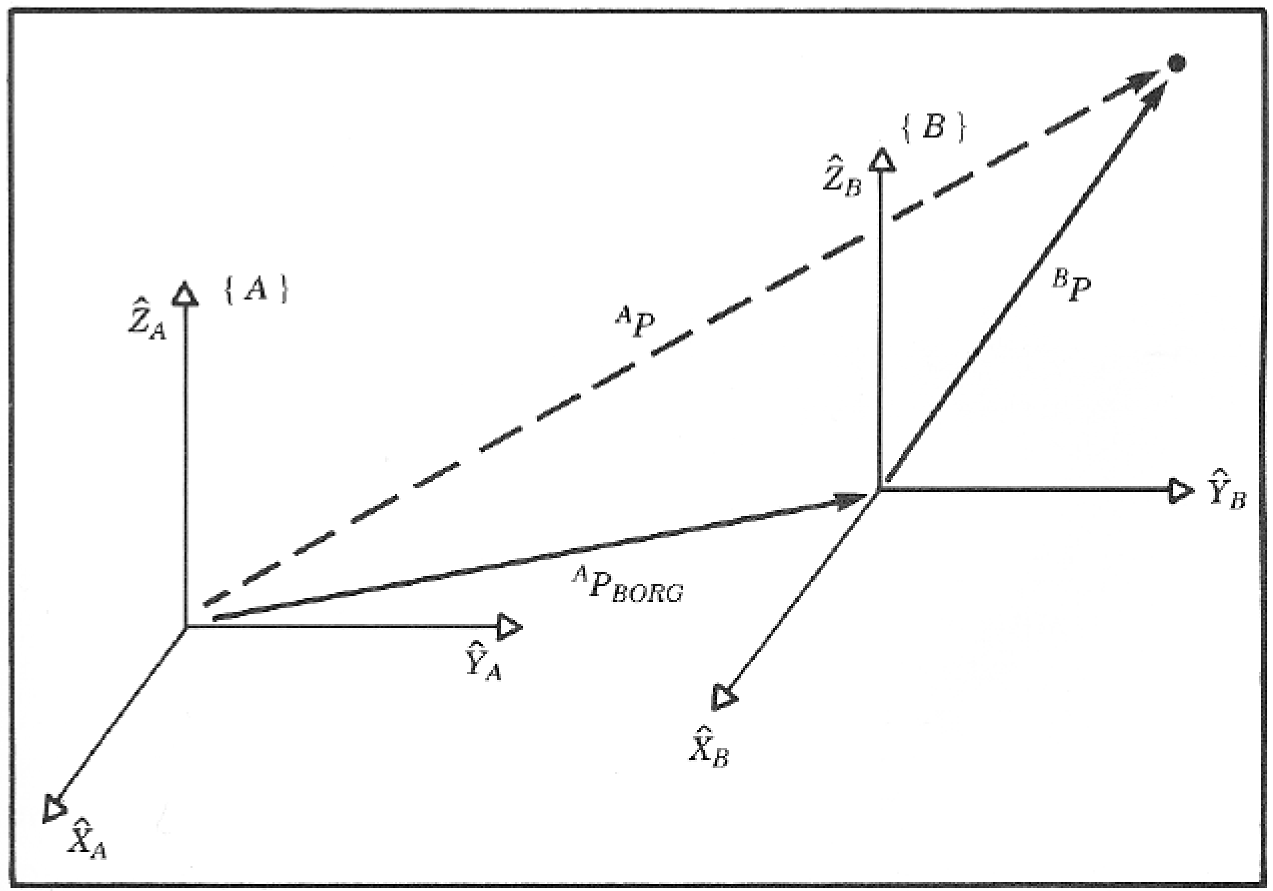
The origin of frame {B} relative to frame {A}



Translation of Frame

- Translation of Frame **B** with respect to Frame **A**

$$^{A}P = {}^{B}P + {}^{A}P_{BORG}$$

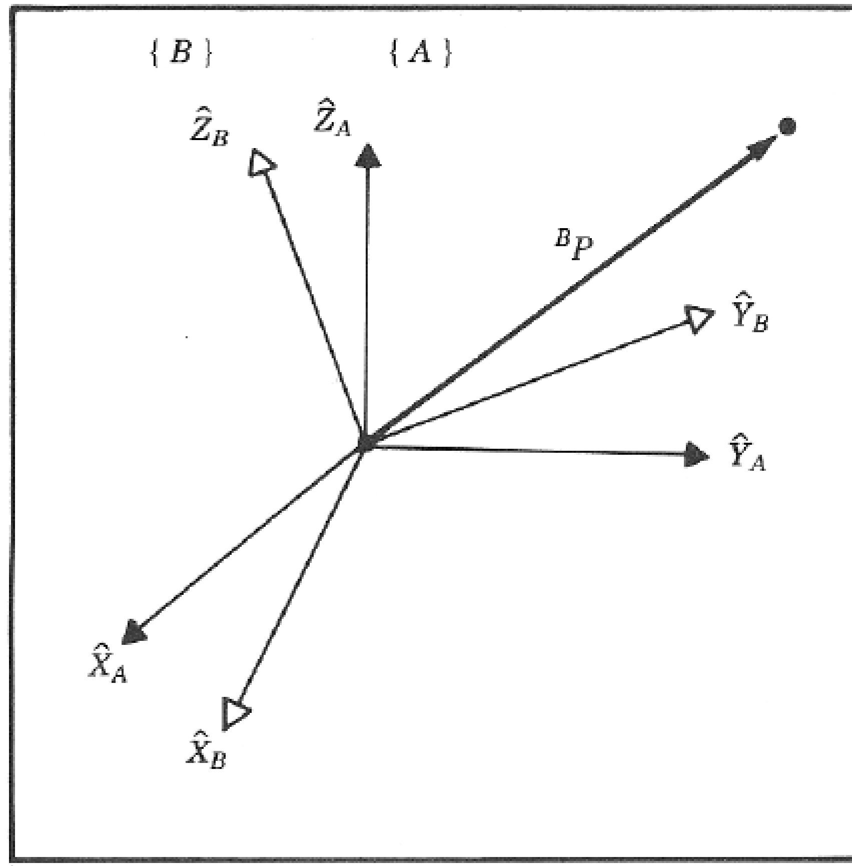


Rotation of a Frame

- Rotation of Frame **B** with respect to Frame **A**

$$ {}^{A}P = {}^{A}_{B}R \; {}^{B}P $$

$$ {}^{B}P = {}^{B}_{A}R \; {}^{A}P $$



Inverse a Rotation Frame

- Given
  - Rotation matrix from Frame **B** with respect to Frame **A** ${}^{A}_{B}R$
- Calculate
  - Rotation matrix from Frame **A** with respect to Frame **B** ${}^{B}_{A}R$

Inverse a Rotation Frame

$$ {}^{A}P = {}^{A}_{B}R \; {}^{B}P $$

$$ {}^{A}_{B}R^{-1} \, {}^{A}P = {}^{A}_{B}R^{-1} \, {}^{A}_{B}R \; {}^{B}P $$

$$ {}^{A}_{B}R^{-1} \, {}^{A}_{B}R = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ P = IP $$

$$ {}^{A}_{B}R^{-1} \, {}^{A}P = {}^{A}_{B}R^{-1} \, {}^{A}_{B}R \; {}^{B}P = I \, {}^{B}P = {}^{B}P $$

$$ {}^{B}P = {}^{A}_{B}R^{-1} \, {}^{A}P $$

$$ {}^{B}P = {}^{B}_{A}R \; {}^{A}P $$

$$ {}^{B}_{A}R = {}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T} $$

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T} $$

Orthogonal Coordinate system

Rotation Frames - Example

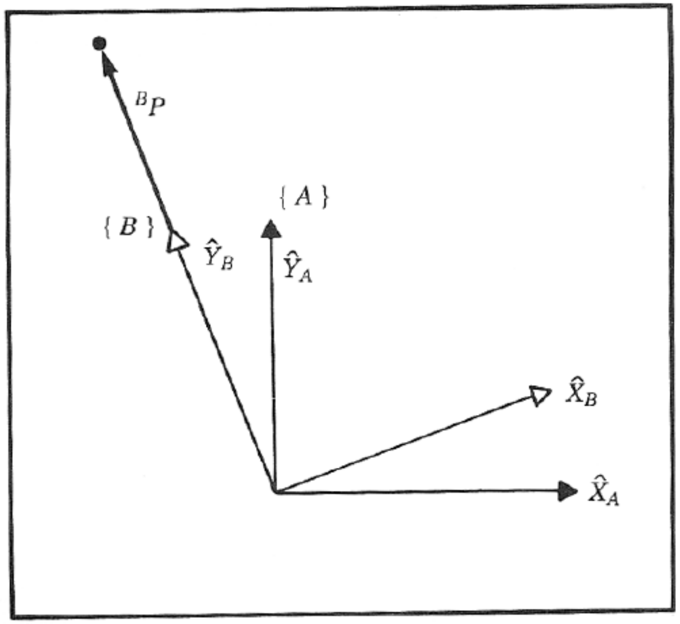
- Given

$$^{B}P = \begin{bmatrix} 0 \\ ^{B}p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\theta = 30^o$$

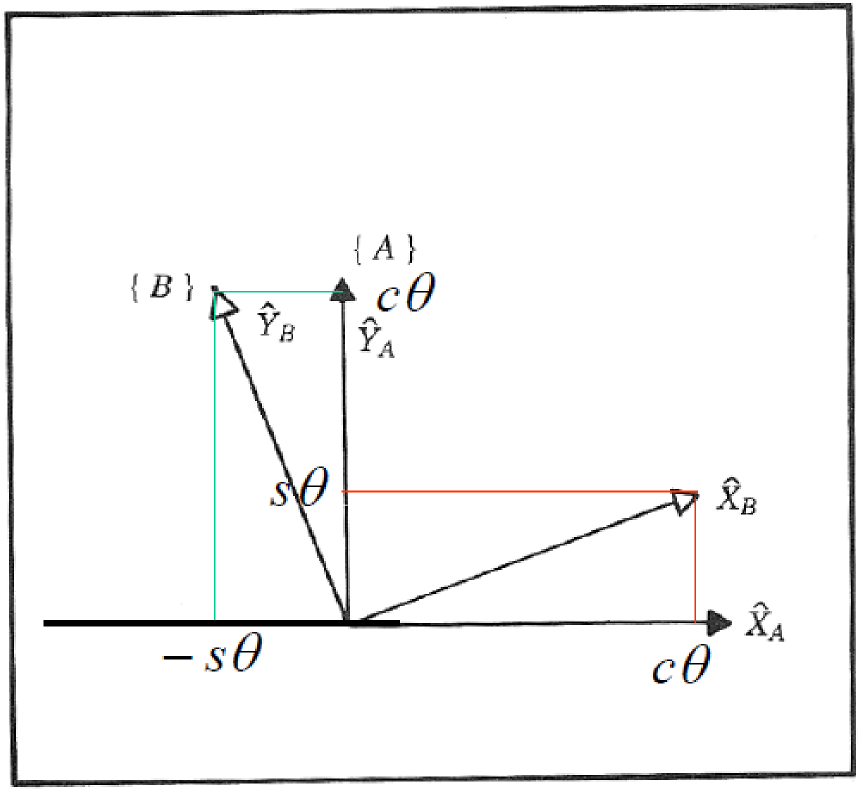
- Calculate ^{A}P

- Solution $^{A}P = {}^{A}_{B}R \; ^{B}P$



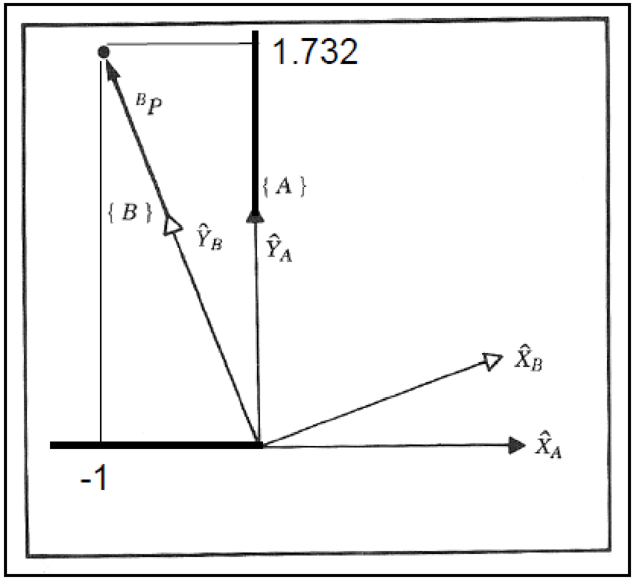
Rotation Frames - Example

$$ {}^{A}_{B}R = [{}^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B] = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



Rotation Frames - Example

$$
{}^{A}P = {}^{A}_{B}R\ {}^{B}P = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ {}^{B}p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 0.000 \\ 2.000 \\ 0.000 \end{bmatrix} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}
$$



Rotation about X, Y, and Z-axis

- Rotation matrices

$$R_X(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \quad R_Y(\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \quad R_Z(\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

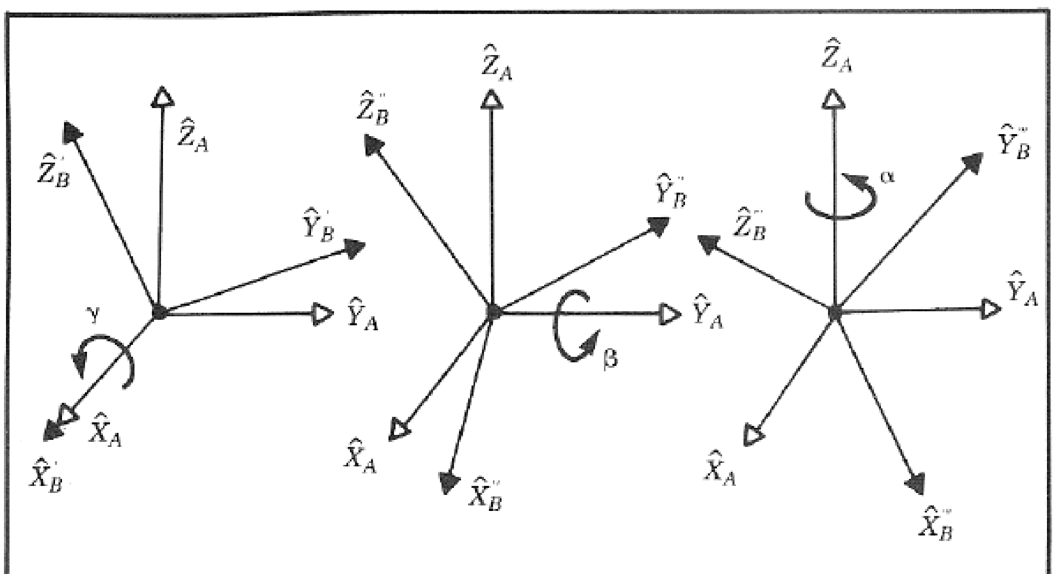
- Rotation about
  - A fixed reference frame
  - A moving reference frame – **Euler Angle**

Rotation about a Fixed Frame

- Each rotation of Frame **B** takes place about an axis in Frame **A**

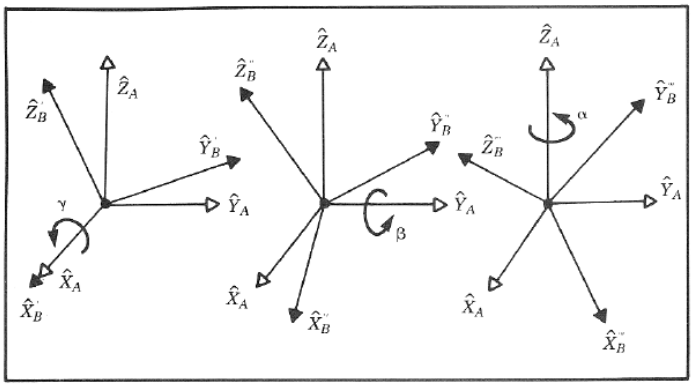
- Rotate frame {B} about $\hat{X}_A$ by an angle γ
- Rotate frame {B} about $\hat{Y}_A$ by an angle β
- Rotate frame {B} about $\hat{Z}_A$ by an angle α

} **Fixed Angles**



Rotation about a Fixed Frame

- Compute rotation with respect to (World) Frame **A**



3 2 1

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

Rotation about a Fixed Frame

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

$$ \beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}) \qquad \text{for} \quad -90^\circ \le \beta \le 90^\circ $$

$$ \alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta) $$

$$ \gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta) $$

- Special case

$$ \beta = \pm 90^\circ $$

$$ \alpha = 0 $$

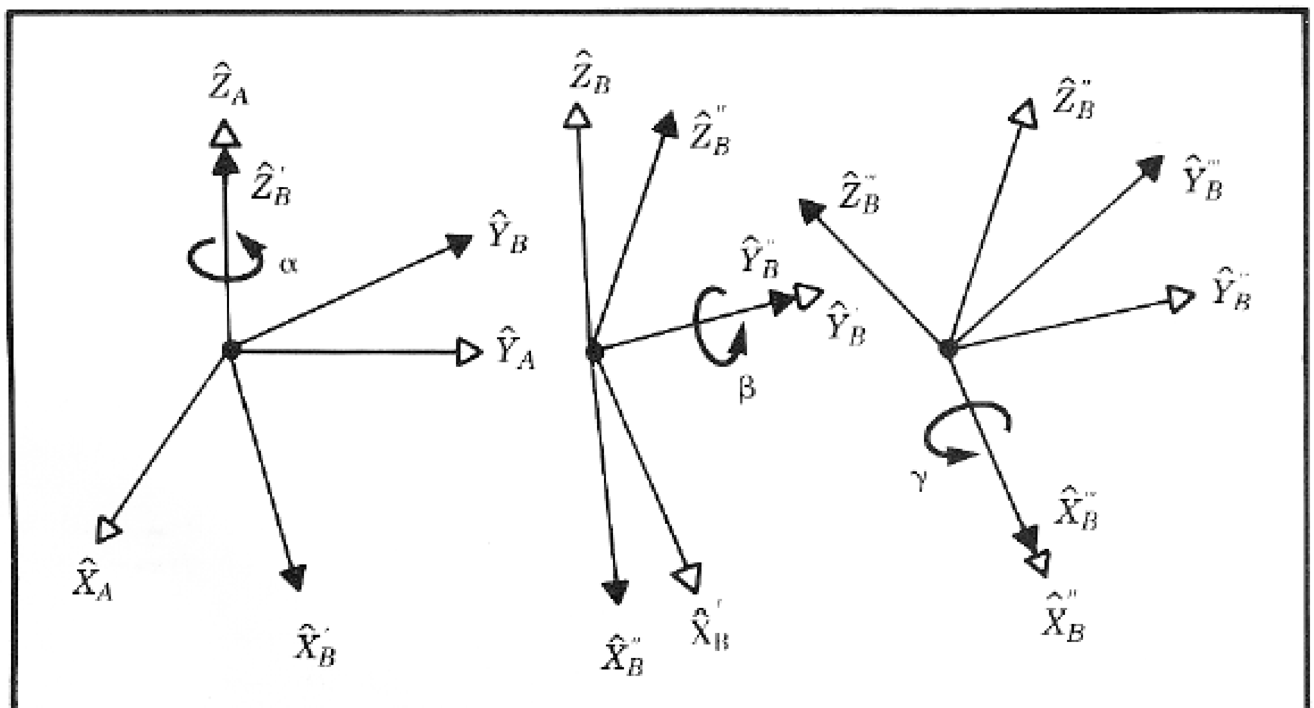
$$ \gamma = \pm \text{Atan2}(r_{12}, r_{22}) $$

Rotation about a Moving Frame – Euler Angle

- Each rotation of Frame **B** takes place about an axis in Frame **B**

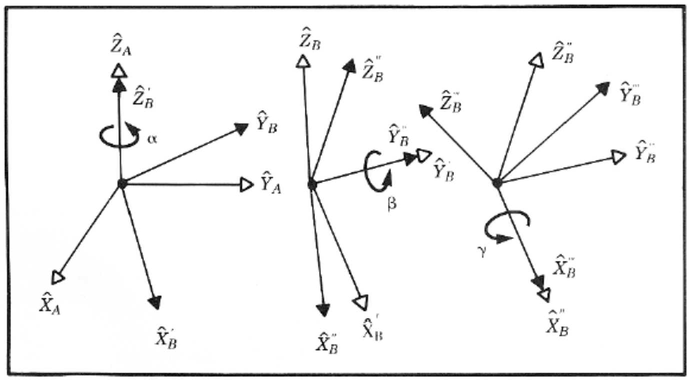
- Rotate frame {B} about $\hat{Z}_A$ by an angle α
- Rotate frame {B} about $\hat{Y}_B$ by an angle β
- Rotate frame {B} about $\hat{X}_B$ by an angle γ

} **Euler Angles**



Rotation about a Moving Frame – Euler Angle

- Compute rotation with respect to (local) Frame **B**



$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

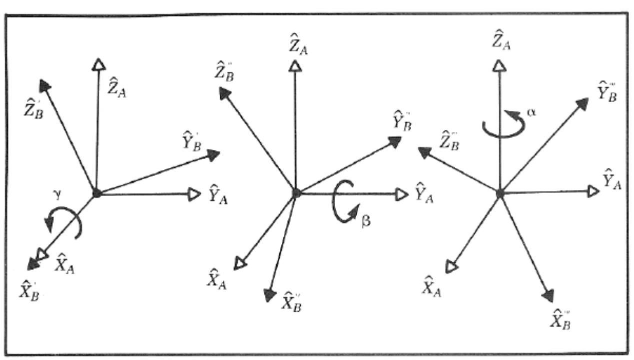
Fixed Angle VS Euler Angle

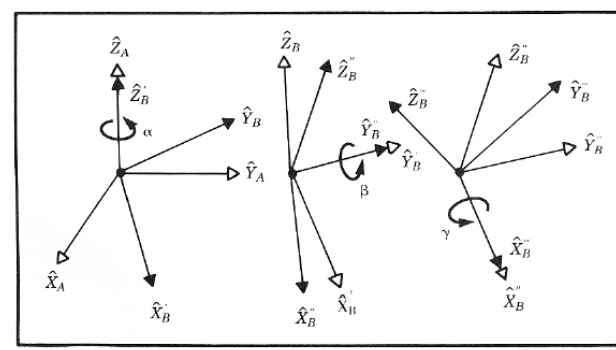
- Same result, but different operation order

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) $$

Fixed Angles
XYZ

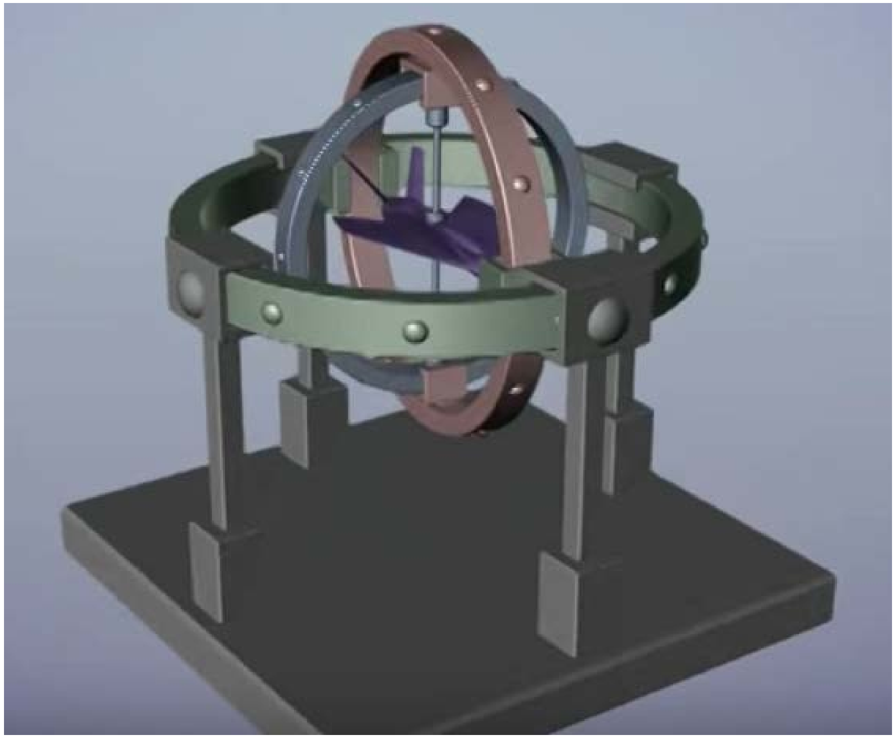
Euler Angles
ZYX

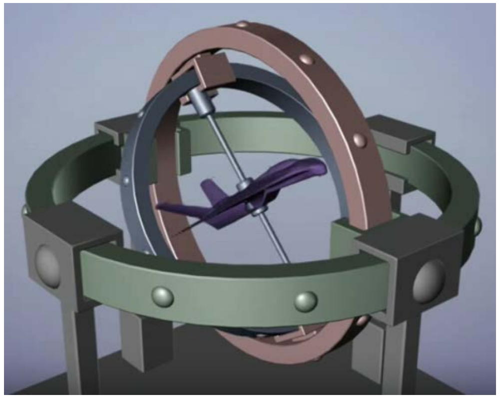


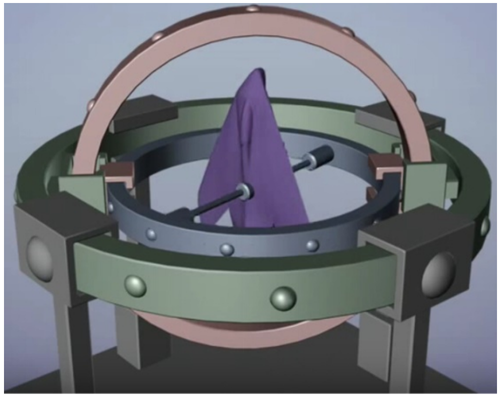


Gimbal Lock

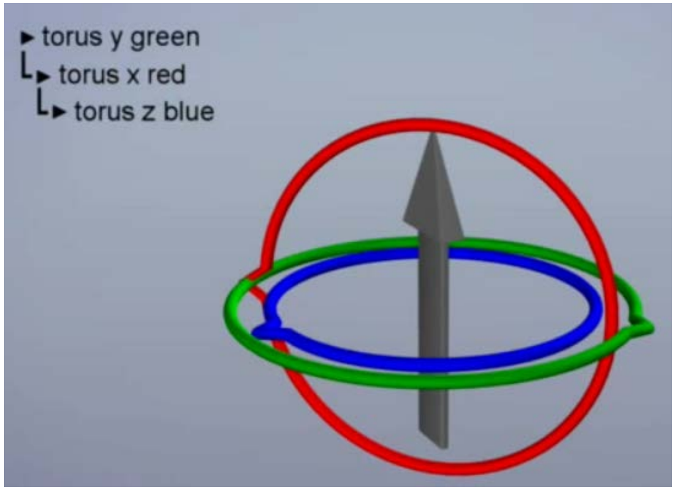
- Because rotations are performed in orders about moving axis
  - Previous operation affects the next operation

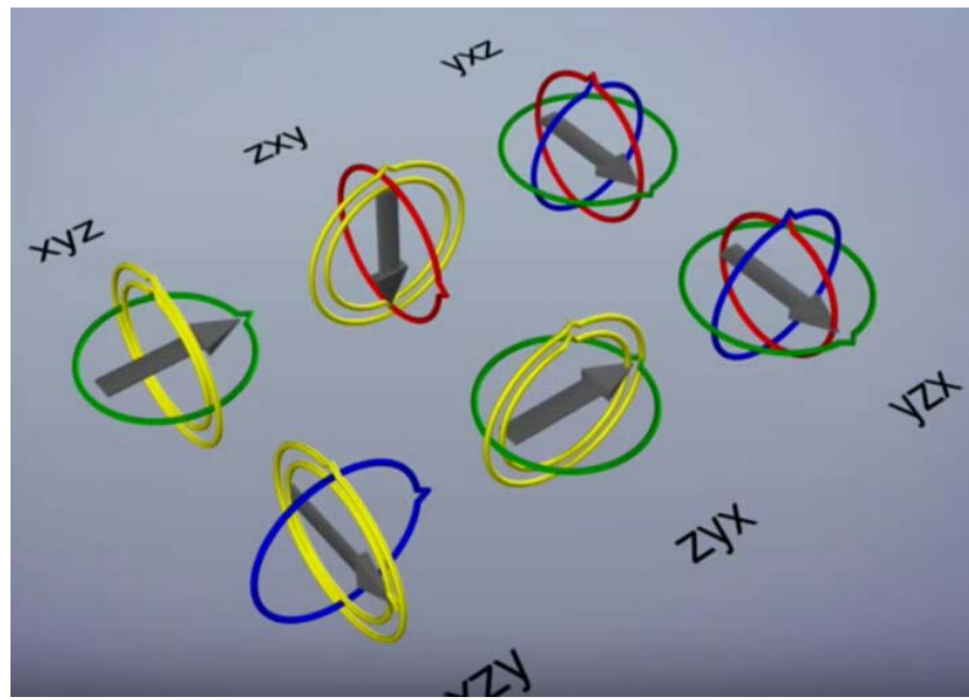






Gimbal Lock





Gimbal Lock – Singularity Example

- Singularities
  - Multiple Euler Angles map to one rotation (Gimbal Lock)
- Let's say this is our convention:

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Lets set $\beta = 0$

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Multiplying through, we get:

$$R = \begin{bmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\sin\gamma - \sin\alpha\cos\gamma & 0 \\ \sin\alpha\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Simplify:

$$R = \begin{bmatrix} \cos(\alpha+\gamma) & -\sin(\alpha+\gamma) & 0 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

α and γ do the same thing! We have lost a degree of freedom!

Homogeneous Transformation

General Frame = Translation + Rotation

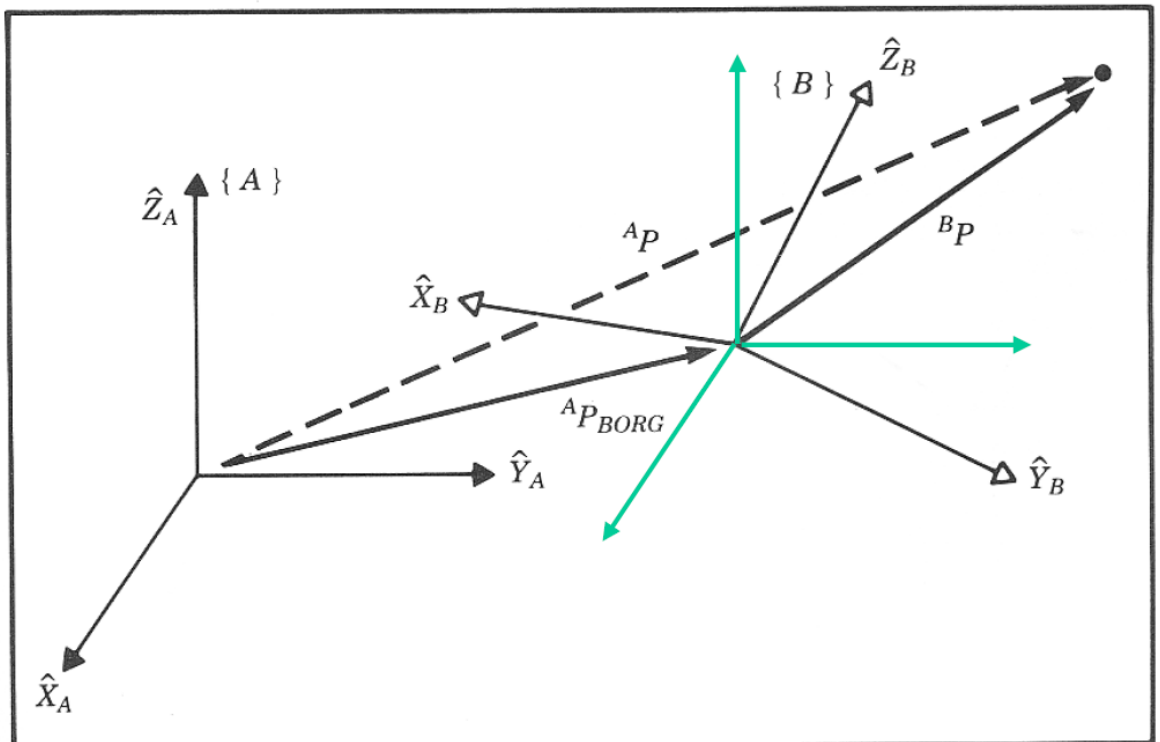
- Frame **B** with respect to Frame **A**

$$\{B\} = \{{}^{A}_{B}R, {}^{A}P_{BORG}\}$$

$${}^{A}P = {}^{A}_{B}P + {}^{A}P_{BORG}$$

$${}^{A}P = {}^{A}_{B}R \; {}^{B}P + {}^{A}P_{BORG}$$

$${}^{A}P = {}^{A}_{B}T \, {}^{B}P$$



Homogeneous Transformation

- Translation and Rotation in a single matrix

$$ {}^{A}P = {}^{A}_{B}R\ {}^{B}P + {}^{A}P_{BORG} $$

$$ {}^{A}P = {}^{A}_{B}T\ {}^{B}P $$

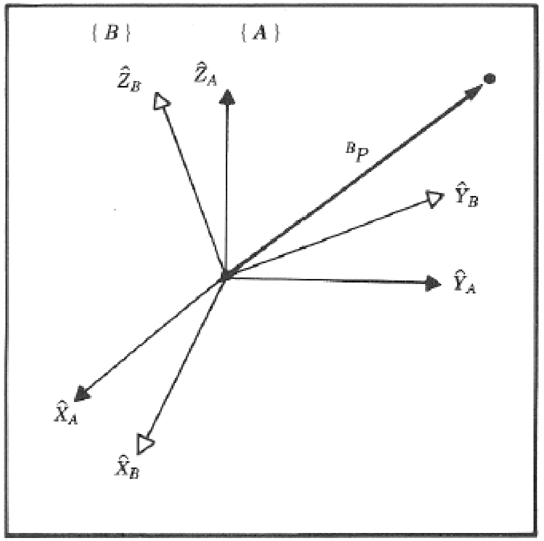
$$ \begin{bmatrix} {}^{A}P \\ \hline 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ \hline 1 \end{bmatrix} $$

- Which one is performed first?

First Rotation, Then Translation

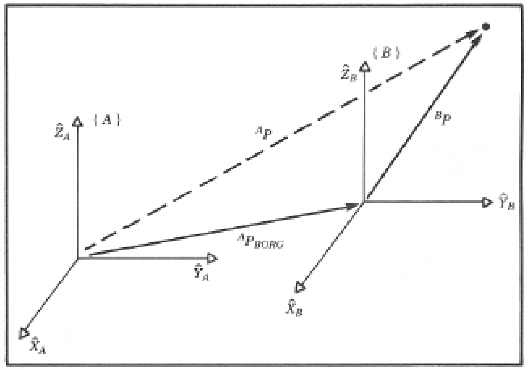
Rotation

$$ {}^{A}_{B}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Translation

$$ {}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & {}^{A}P_{BORGx} \\ 0 & 1 & 0 & {}^{A}P_{BORGy} \\ 0 & 0 & 1 & {}^{A}P_{BORGz} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Example

- Given

$$^{B}P = \begin{bmatrix} ^{B}p_x \\ ^{B}p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$$

- Manipulation
  - Rotate Frame **B** about **Z**-axis of Frame **A** by **30** deg, **then**
  - Translate **10** units along **X**-axis and 5 units along **Y**-axis, in Frame **A**
- Find
  - ^{A}P in Frame A

Example

$$^{A}P = {}^{A}_{B}T\,{}^{B}P = \begin{bmatrix} ^{A}P \\ \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{B}P \\ \\ 1 \end{bmatrix}$$

$$^{A}P = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.0 \\ 1 \end{bmatrix}$$

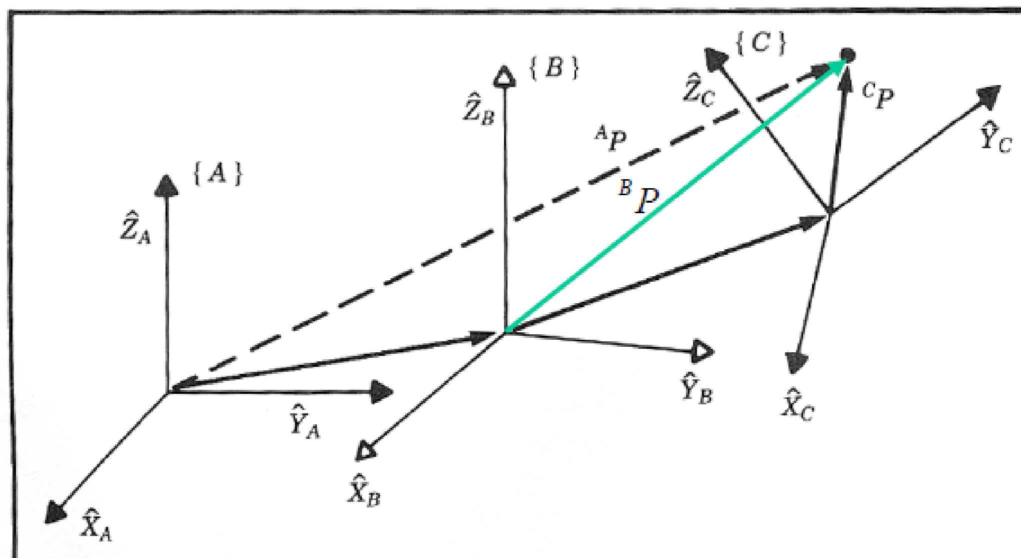
Compound Transformation

- Given
  Vector ${}^{C}P$
  Frame {C} is known relative to frame {B} - ${}^{B}_{C}T$
  Frame {B} is known relative to frame {A} - ${}^{A}_{B}T$
- Calculate
  Vector ${}^{A}P$

$${}^{B}P = {}^{B}_{C}T\,{}^{C}P$$

$${}^{A}P = {}^{A}_{B}T\,{}^{B}P$$

$${}^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P$$



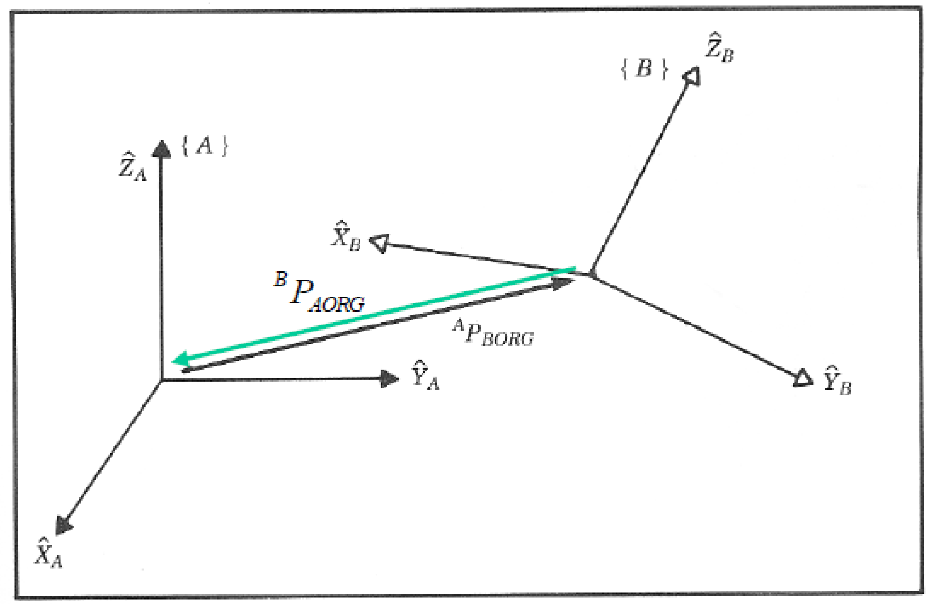
Inverted Transformation

- Given frame {B} relative to frame {A} - ${}^{A}_{B}T \quad ({}^{A}_{B}R, {}^{A}P_{BORG})$
- Calculate frame {A} relative to frame {B} - ${}^{B}_{A}T \quad ({}^{B}_{A}R, {}^{B}P_{AORG})$

$$ {}^{B}_{A}R = {}^{A}_{B}R^{T} $$

$$ {}^{B}P_{AORG} = -{}^{B}_{A}R\,{}^{A}P_{BORG} = -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} $$

$$ {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$



Note: $\quad {}^{B}_{A}T = {}^{A}_{B}T^{-1}$

Example

- Given

Description of frame {B} relative to frame {A} - ${}^{A}_{B}\hat{T}$ $\quad ({}^{A}_{B}R, {}^{A}P_{BORG})$

- Manipulation
  - Rotate Frame B about Z-axis of Frame A by 30 deg, **then**
  - Translate 4 units along X-axis and 3 units along Y-axis, in Frame A
- Find

Homogeneous Transform ${}^{B}_{A}T$ $({}^{B}_{A}R, {}^{B}P_{AORG})$

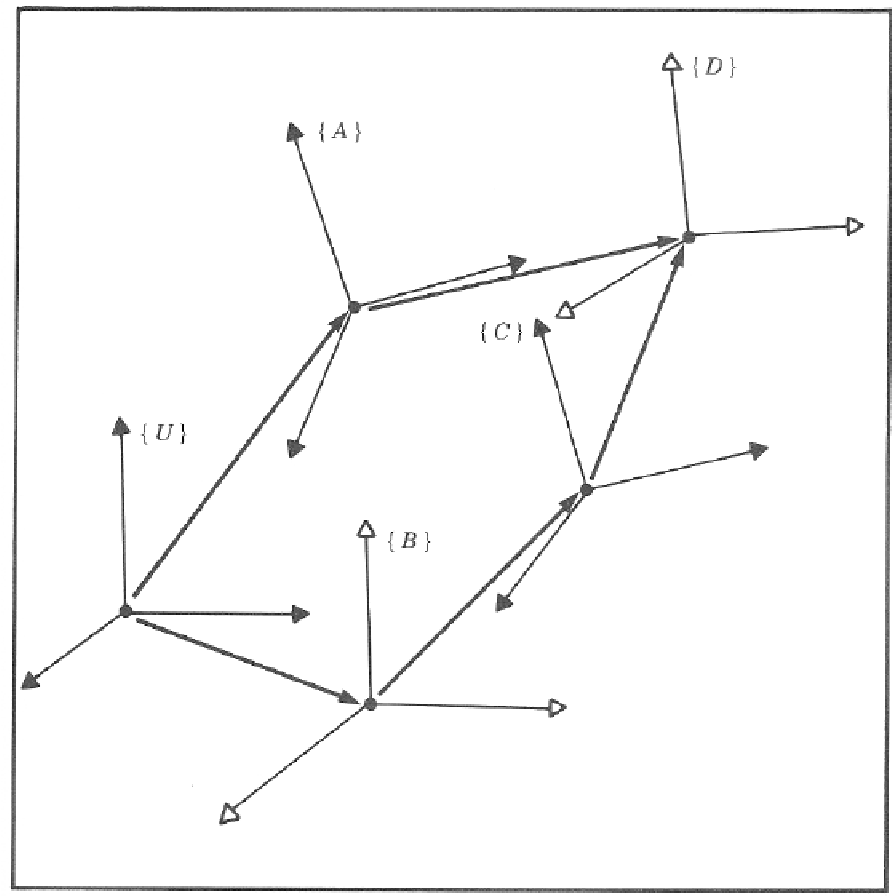
Example

$$
{}^{A}_{B}T = \left[\begin{array}{ccc|c} c\theta & -s\theta & 0 & {}^{A}P_{BORGx} \\ s\theta & c\theta & 0 & {}^{A}P_{BORGy} \\ 0 & 0 & 1 & {}^{A}P_{BORGz} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0.866 & -0.500 & 0.000 & 4.000 \\ 0.500 & 0.866 & 0.000 & 3.000 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
{}^{B}_{A}T = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0.866 & 0.500 & 0.000 & -4.964 \\ -0.500 & 0.866 & 0.000 & -0.598 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Transformation Equations

- Given $\,^{U}_{A}T,\ ^{A}_{D}T,\ ^{U}_{B}T,\ ^{C}_{D}T$
- Calculate $\,^{B}_{C}T$



$$^{U}_{D}T = ^{U}_{A}T\, ^{A}_{D}T$$

$$^{U}_{D}T = ^{U}_{B}T\, ^{B}_{C}T\, ^{C}_{D}T$$

$$^{U}_{A}T\, ^{A}_{D}T = ^{U}_{B}T\, ^{B}_{C}T\, ^{C}_{D}T$$

$$^{U}_{B}T^{-1}\, ^{U}_{A}T\, ^{A}_{D}T\, ^{C}_{D}T^{-1} = ^{U}_{B}T^{-1}\, ^{U}_{B}T\, ^{B}_{C}T\, ^{C}_{D}T\, ^{C}_{D}T^{-1}$$

$$^{B}_{C}T = ^{U}_{B}T^{-1}\, ^{U}_{A}T\, ^{A}_{D}T\, ^{C}_{D}T^{-1}$$

Quaternion

Quaternions

- Generalizations of complex numbers

$$\tilde{q} = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$$

- Identities

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

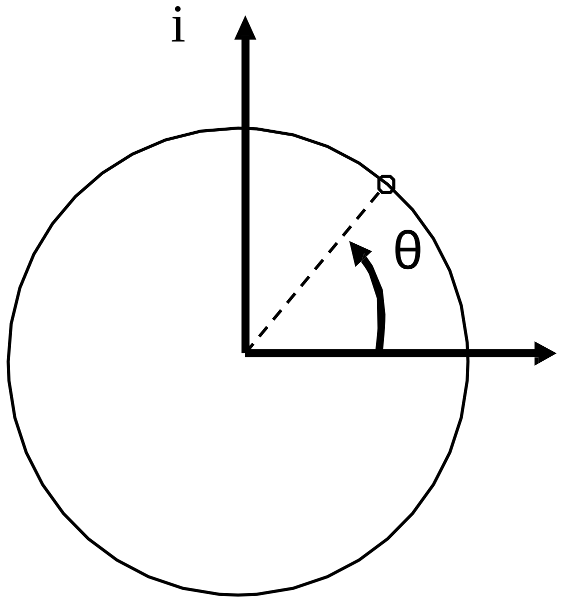
×	1	*i*	*j*	*k*
1	1	*i*	*j*	*k*
i	*i*	−1	*k*	−*j*
j	*j*	−*k*	−1	*i*
k	*k*	*j*	−*i*	−1

Intuition from Complex Numbers

- Use a second "imaginary" dimension
- Permits manipulation of rotations like a vector

$$\tilde{q} = q_1 + q_2\mathbf{i} + q_3\mathbf{i} + q_4\mathbf{k}$$

Rotation about 3 axes



Notation

- 4-tuples

$$\tilde{q} = (q_1, q_2, q_3, q_4)$$

- Hyper-complex number

$$\tilde{q} = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$$

- Real + Imaginary

$$\tilde{q} = q + \vec{q}$$

- Ordered doublet

$$\tilde{q} = (q, \vec{q})$$

- Exponential

$$\tilde{q} = e^{\frac{1}{2}\theta\vec{w}}$$

Operation

- Addition

$$\tilde{p} + \tilde{q} = (p_1 + q_1) + (p_2 + q_2)\mathbf{i} + (p_2 + q_2)\mathbf{j} + (p_2 + q_2)\mathbf{k}$$

- Multiplication

$$\tilde{p}\tilde{q} = (p + \vec{p})(q + \vec{q})$$

$$\tilde{p}\tilde{q} = (pq - \vec{p} \cdot \vec{q}) + (p\vec{q} + q\vec{p} + \vec{p} \times \vec{q})$$

$$\tilde{p}\tilde{q} \neq \tilde{q}\tilde{p}$$

Non-commutative

Operation

- Conjugate – just like a complex number

$$\tilde{q}^* = q - \vec{q}$$

- Dot product

$$\tilde{q} \cdot \tilde{q} = pq + \vec{p} \cdot \vec{q}$$

- Norm

$$|q| = \sqrt{\tilde{q} \cdot \tilde{q}}$$

- Product with conjugate $=$ dot product

$$\tilde{q}\tilde{q}^* = qq + \vec{q} \cdot \vec{q} = \tilde{q} \cdot \tilde{q}$$

Another way to compute norm

Operation

- Inverse

$$\frac{\tilde{q}\tilde{q}^*}{|\tilde{q}|^2} = 1$$

- Therefore

$$\tilde{q}^{-1} = \frac{\tilde{q}^*}{|\tilde{q}|^2} = 1$$

- Note that for unit quaternion,

$$\tilde{q}\tilde{q}^{-1} = \tilde{q}\tilde{q}^* = |\tilde{q}|^2 = 1$$

Rotation of a Vector

- Convert a vector to quaternion

$$\tilde{x} = 0 + \vec{x} \qquad \text{given} \quad \vec{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

- Rotation axis + Rotation angle

$$\boxed{\tilde{q} = e^{\frac{1}{2}\theta\vec{w}}} \Rightarrow \tilde{q} = \cos\frac{\theta}{2} + \vec{w}\sin\frac{\theta}{2}$$

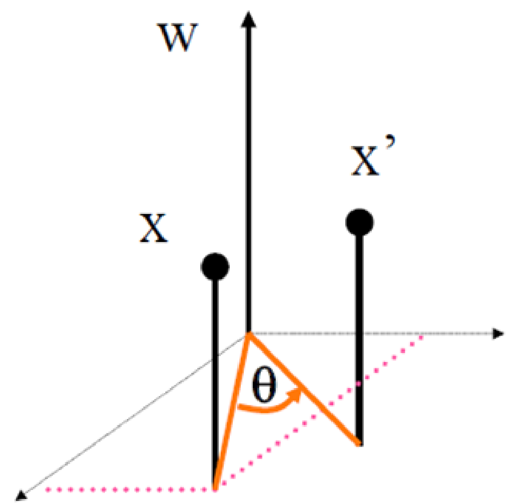
polar decomposition

- Rotating a vector – sandwich

$$\tilde{x}' = \tilde{q}\tilde{x}\tilde{q}^*$$

- Composite rotation

$$\tilde{x}'' = \tilde{p}\tilde{x}'\tilde{p}^* = \tilde{p}\tilde{q}\tilde{x}\tilde{q}^*\tilde{p}^*$$



Quaternion to Rotation Matrix

- For a quaternion

$$\tilde{q} = q_1 + q_2\mathbf{i} + q_3\mathbf{i} + q_4\mathbf{k}$$

- Rotation matrix

$$R = \begin{bmatrix} 2(q_1^2 + q_2^2) - 1 & 2(q_2q_3 - q_1q_4) & 2(q_2q_4 + q_1q_3) \\ 2(q_2q_3 + q_1q_4) & 2(q_1^2 + q_3^2) - 1 & 2(q_3q_4 - q_1q_2) \\ 2(q_2q_4 - q_1q_3) & 2(q_1q_2 + q_3q_4) & 2(q_0^2 + q_4^2) - 1 \end{bmatrix}$$

Comparison

- Euler angle
  - Intuitive
  - Rotation and translation are separate
  - Gimbal lock - Transformation is not unique at singularity
- Homogenous Transform
  - Intuitive
  - Rotation + Translation in one shot
  - Not compact
- Quaternion
  - Rotation and translation are separate, but in the same format
  - Fancy math, less intuitive
  - Compact format – efficient for the computation of some problems

Readings