Kinematics

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Kinematics of Serial Robots

- We know how to describe the transformation of a single rigid object w.r.t. a single frame
- If we have many rigid object in serial connection,

how to express and derive their spatial relations?

Overview – Robot Kinematics

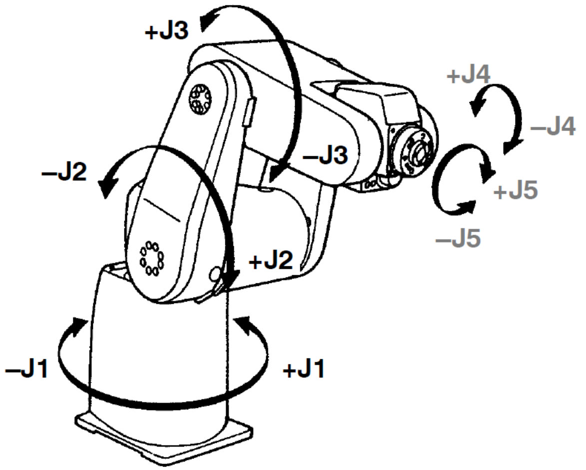
- Forward Kinematics
  - Planar Robotic Systems, Representation of Serial Robots, Open Polygon Model, Denavit-Hartenberg Representation, Singularities

- Inverse Kinematics
  - Kinematic Decoupling, Inverse Position: Geometric Approach, Inverse Orientation

- Kinematics in a Nut Shell

Forward and Inverse Kinematics

- For industrial robots, the main concern is the position and orientation of the end-effector or the attached tool

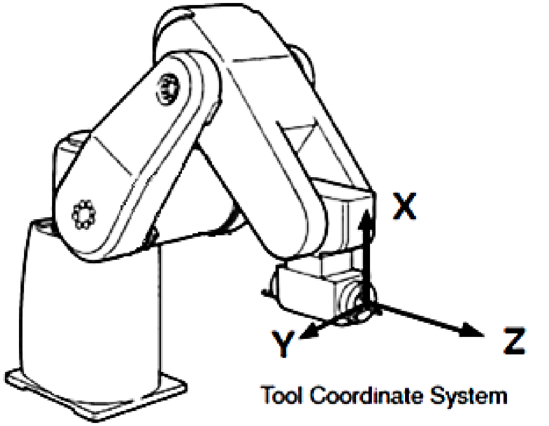


Robot Joint Angles

Forward Kinematics

End Effector Pose

Inverse Kinematics

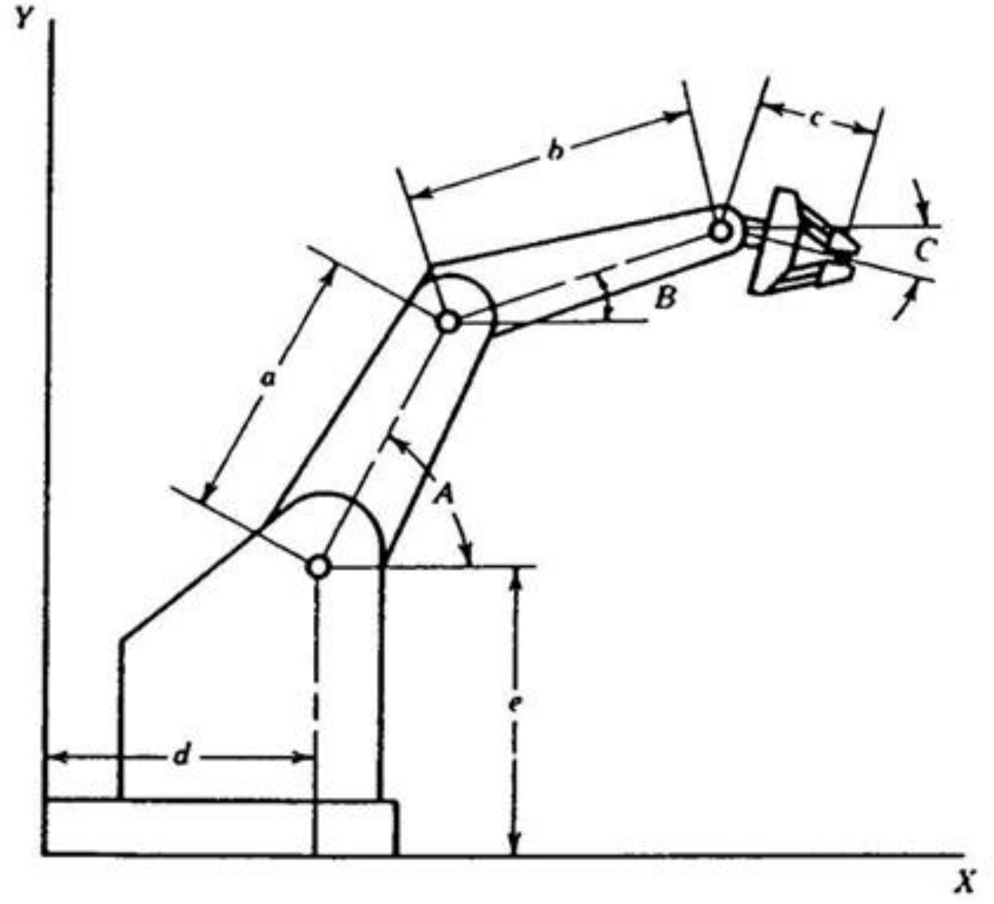


A 2D example

- Tool Center Position (TCP) of a Planar Robotic Manipulator

$$X = d + a\cos A + b\cos B + c\cos C$$

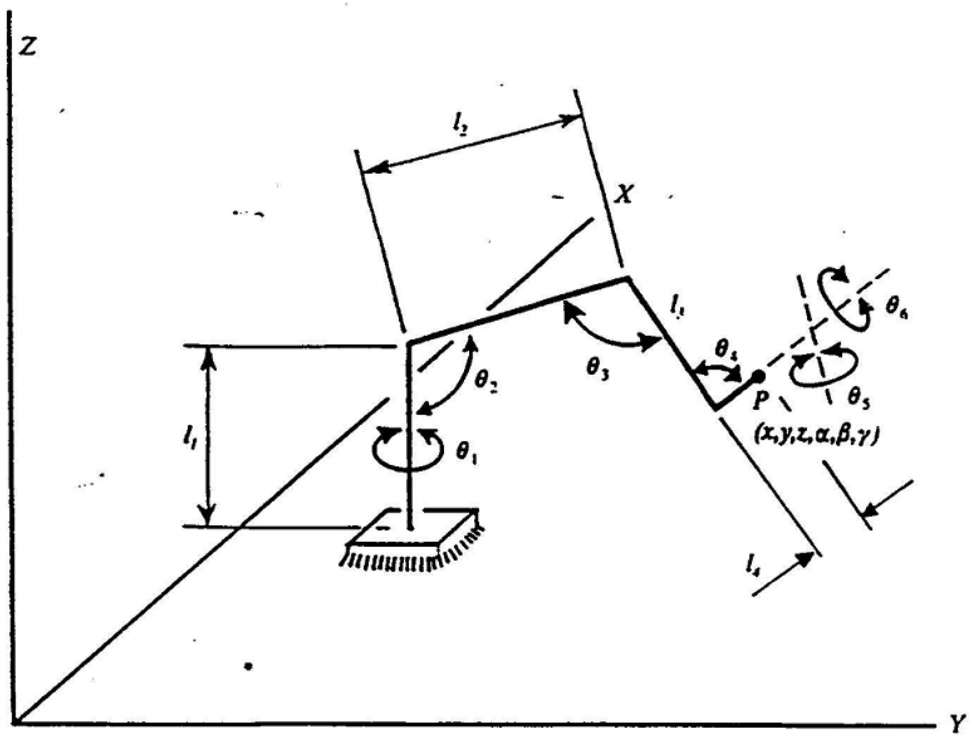
$$Y = e + a\sin A + b\sin B + c\sin C$$



Open Polygon Representation

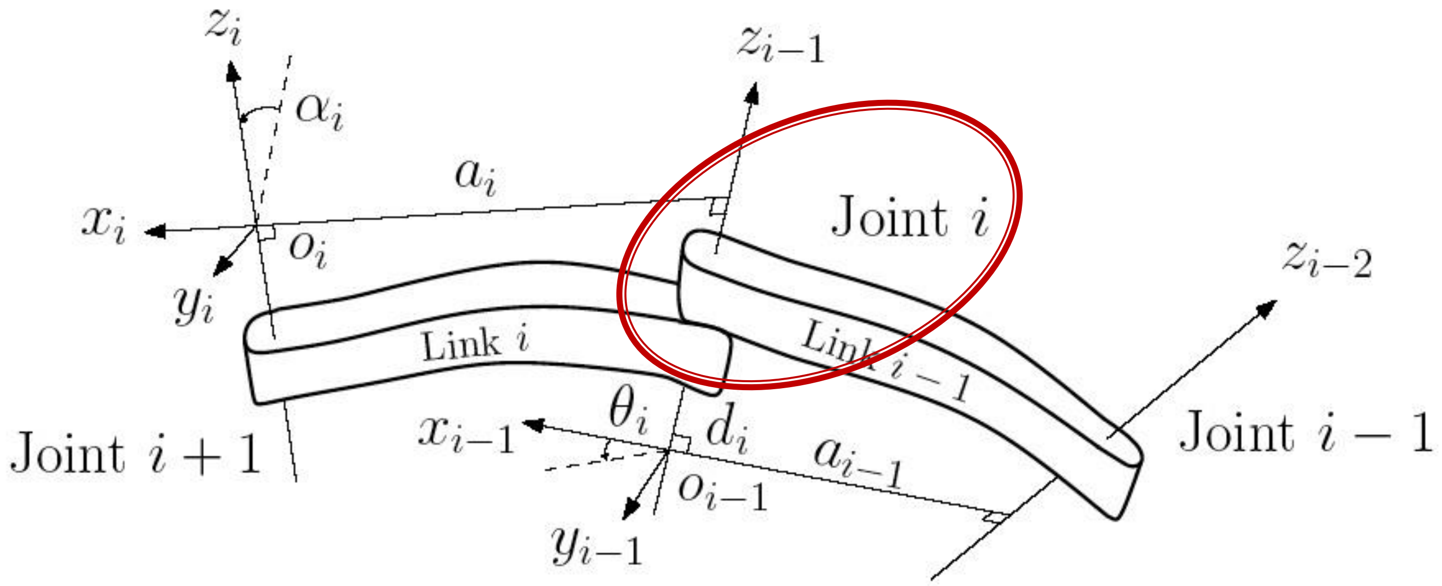
- Homogeneous transformations can be applied to all joints to get the end effector / tool position. However …
  - The transformation matrix depends on how the coordinate systems are set up and how the structural parameters are defined.
  -
  - Hence, how to make sure two people can develop same transformation matrices for the same robot?

A standard method of Kinematics derivation



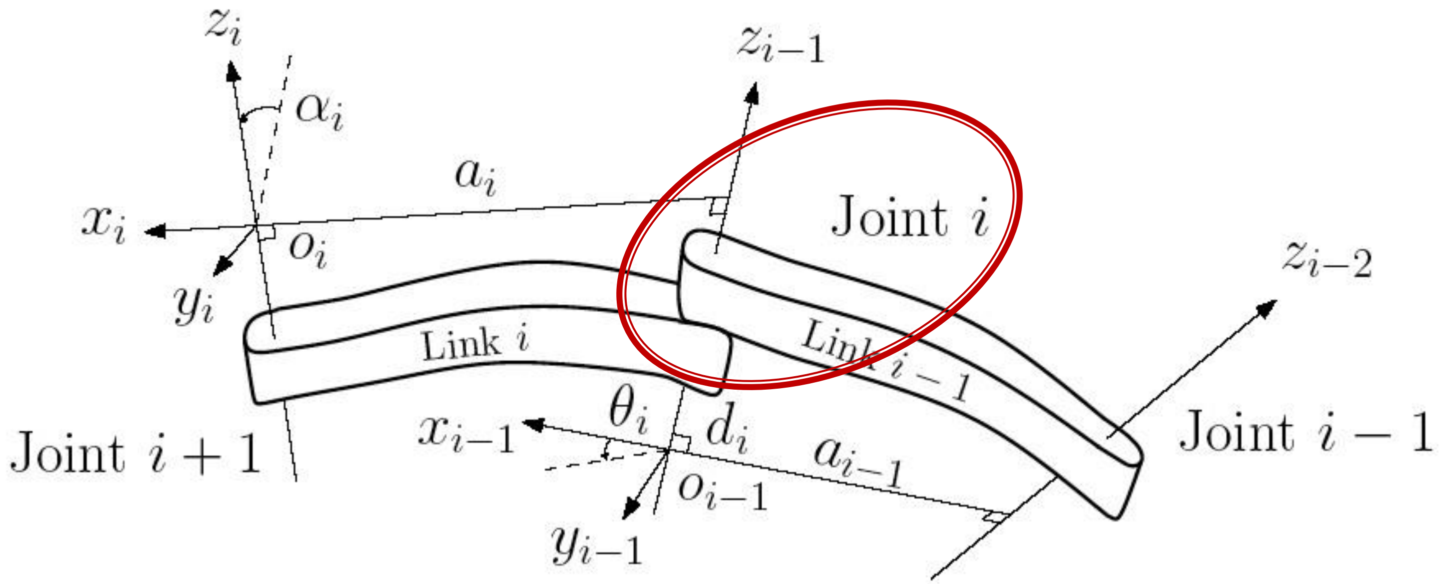
Step 1: Assign local reference frame for each joint (z and x axes)

- Every coordinate frame is established following three rules:
  - The z_{i-1} axis lies along the axis of motion of the ith joint



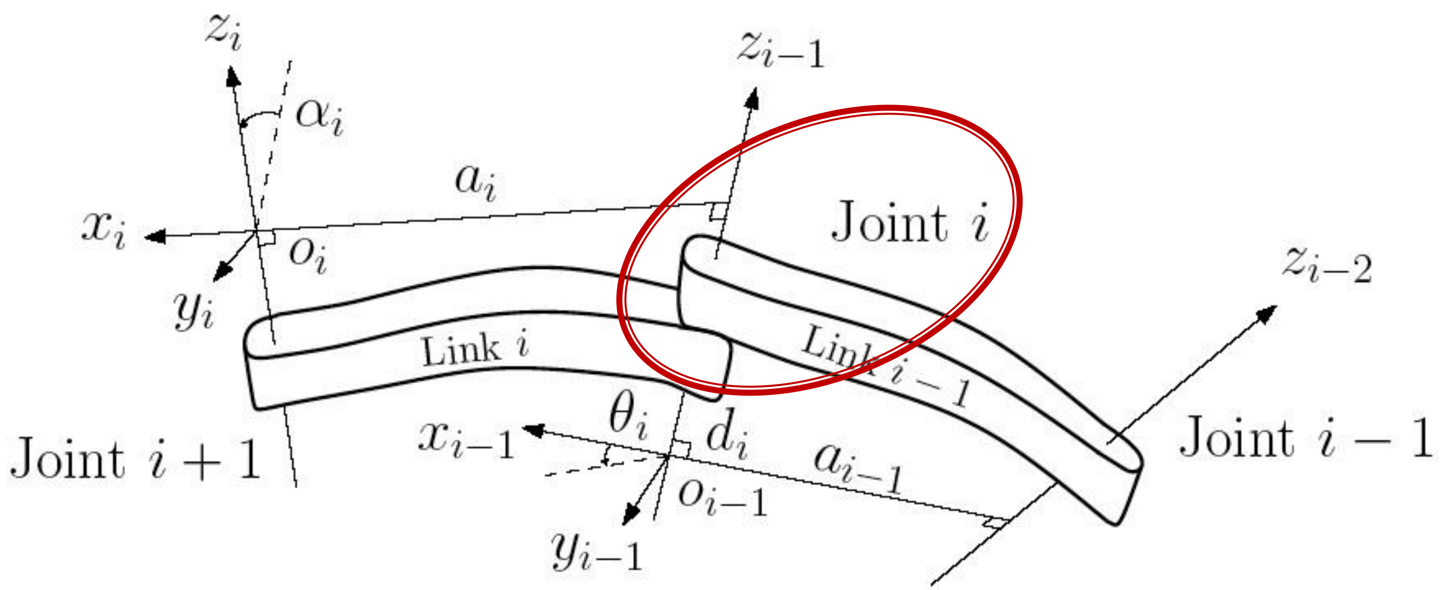
Step 1: Assign local reference frame for each joint (z and x axes)

- Every coordinate frame is established following three rules:
  - The x_i axis is normal to the z_{i-1} axis, and points away from it to the z_i axis



Step 1: Assign local reference frame for each joint (z and x axes)

- Every coordinate frame is established following three rules:
  - The x_i axis forms the common perpendicular between the z_{i-1} and z_i axis



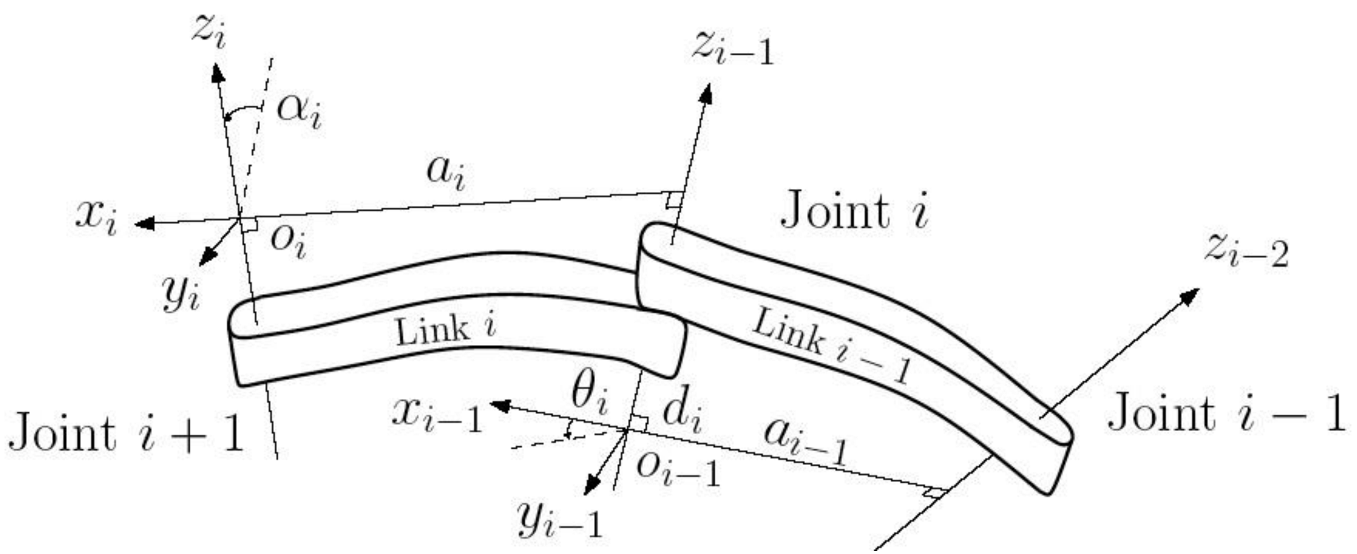
Choices for the base and end-effector frames

- Base Frame
  - You can choose any location for the coordinate frame 0 in the the robot base as long as the z_0 axis is aligned with the first joint
- End-effector Frame
  - The last coordinate frame (nth frame) can be placed anywhere in the tool or end effector, as long as the x_n axis is normal to z_{n-1} axis.

Step 2: Determine the D- H parameters

- Relative pose between rigid bodies
  - Position + Orientation

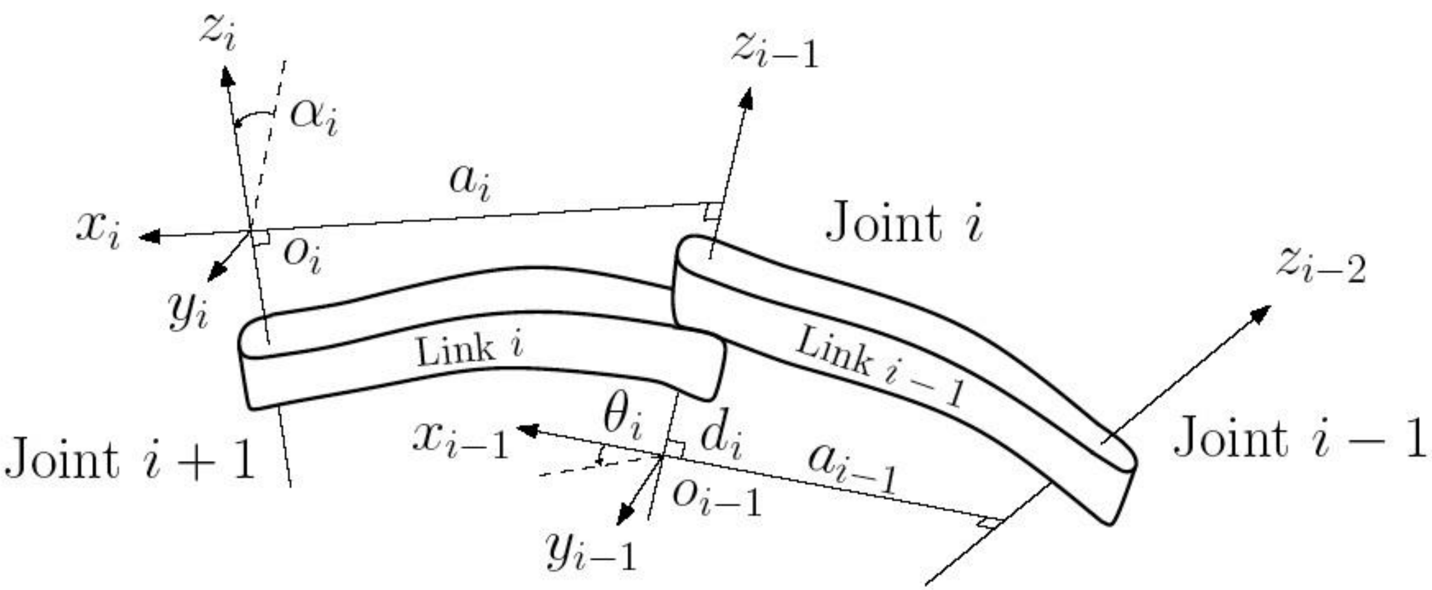
- How many parameters do you need to fully specify their relative pose?



Joint Angle

- θ_i is the joint angle from the x_{i-1} to the x_i axis about the z_{i-1} axis using the right hand rule

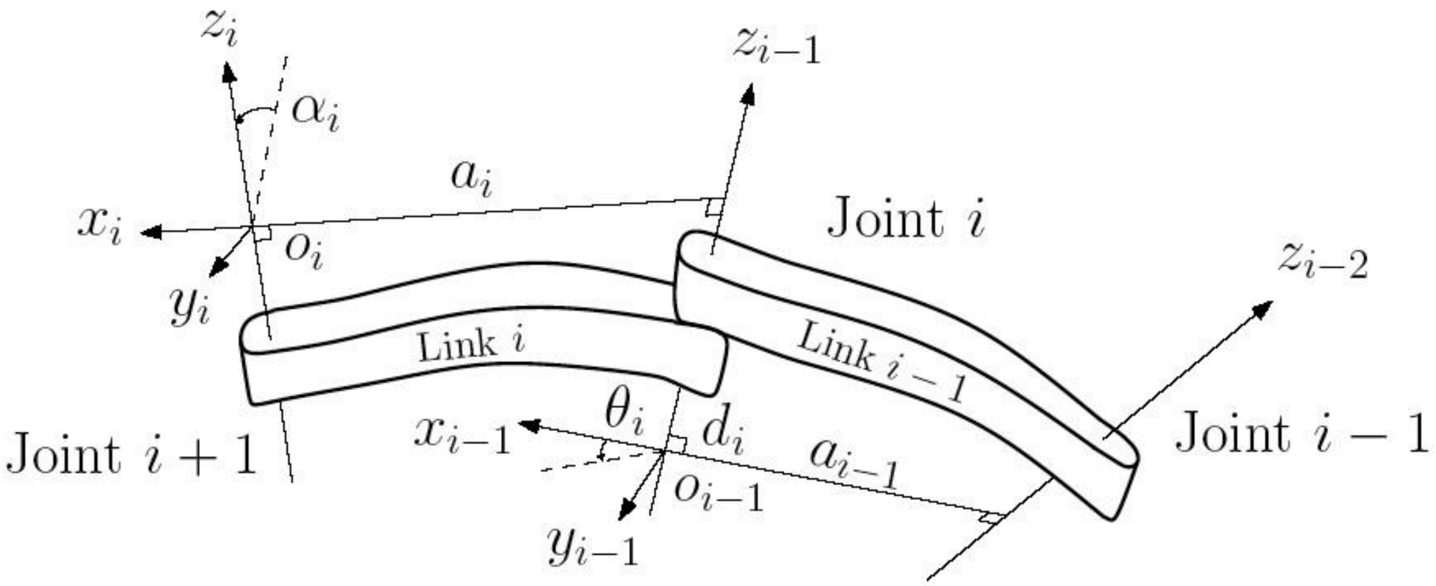
Angle between two X-axes



Link offset

- d_i is the offset distance from the origin of the $(i-1)$th coordinate frame to the intersection of the z_{i-1} axis with the x_i axis along the z_{i-1} axis.

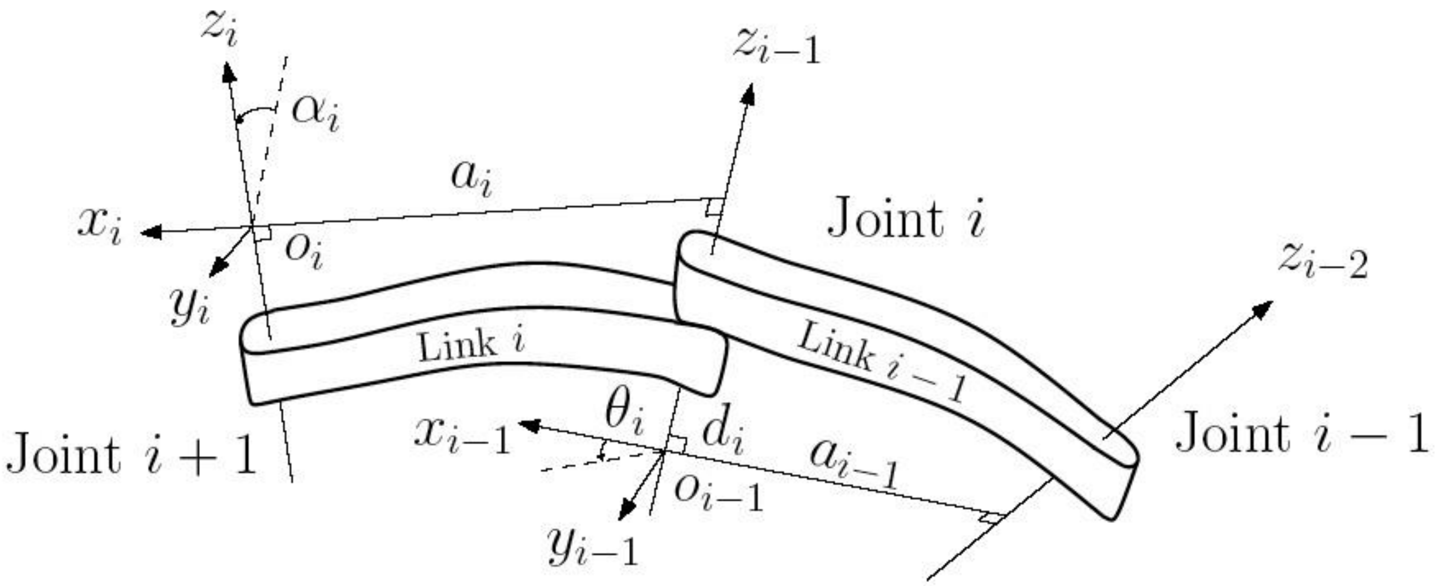
Distance between two X-axes



Link Length

- a_i is the distance from the intersection of the z_{i-1} axis with the x_i axis to the origin of the ith frame along the x_i axis (the shortest distance between the z_{i-1} and z_i axes).

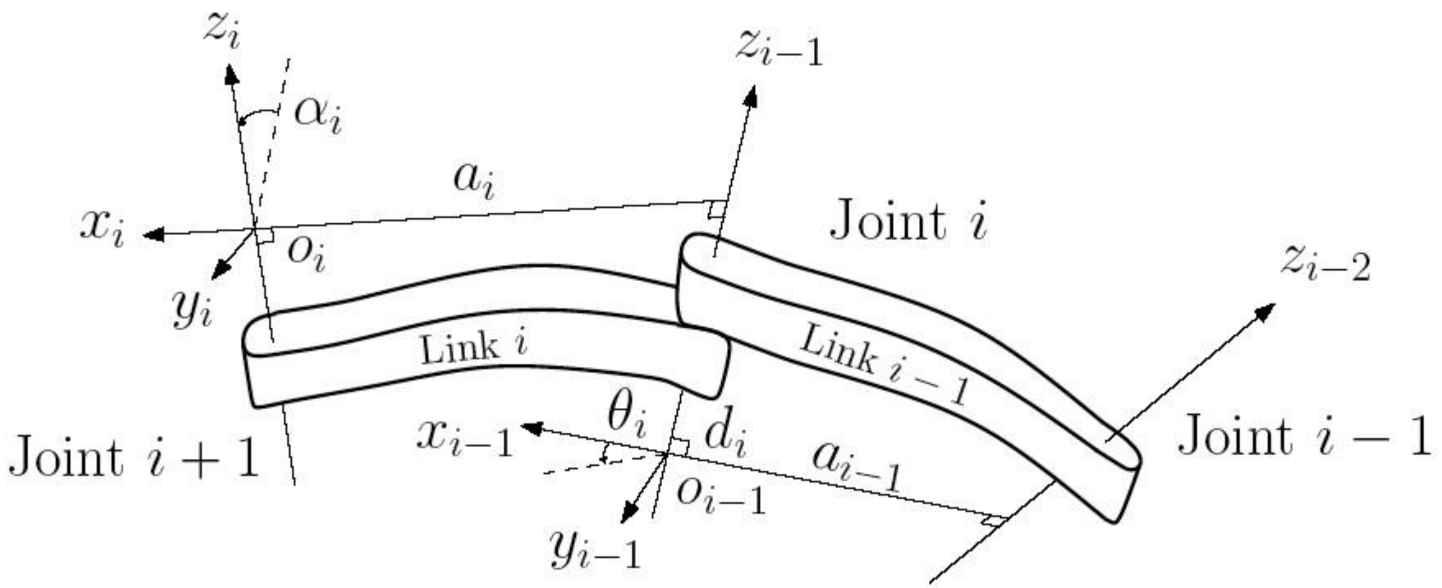
Distance between two Z-axes



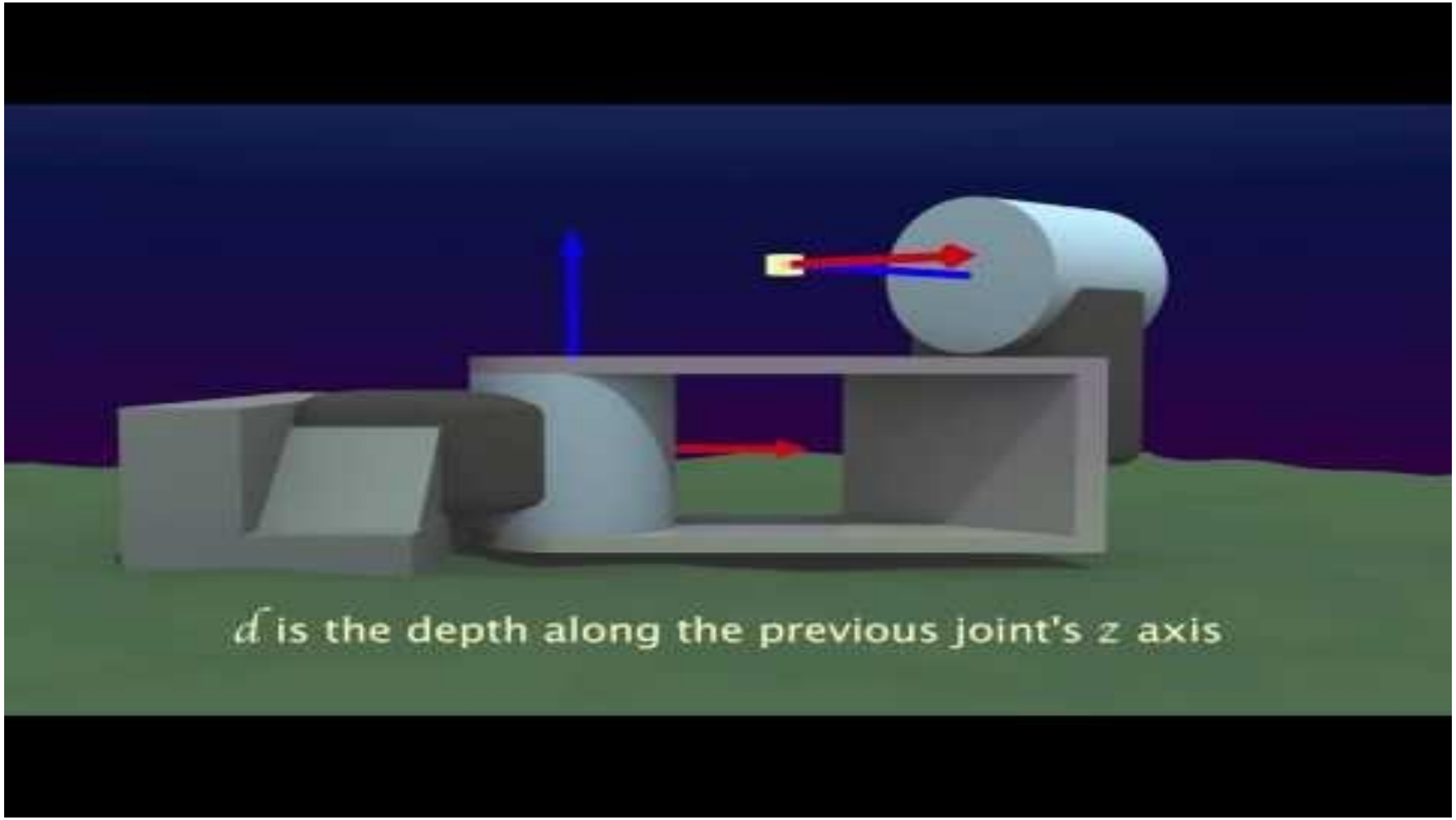
Link Twist

- α_i is the twisted angle from the z_{i-1} axis to the z_i axis about the x_i axis (using the right-hand rule).

Angle between two Z-axes



DH parameters



Step 3: Specify the transformation matrix

$$A_{i-1}^{i} = Rot(z,\theta_i)\cdot Trans(0,0,d_i)\cdot Trans(a_i,0,0)\cdot Rot(x,\alpha_i)$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Complete transformation from base frame to tool frame:

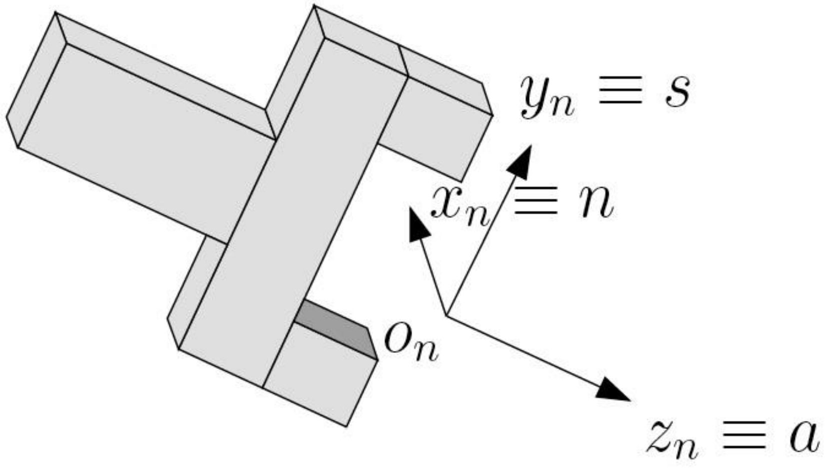
$$T_0^6 = A_0^1 A_1^2 A_2^3 A_3^4 A_4^5 A_5^6$$

Vectors in the Transformation Matrix

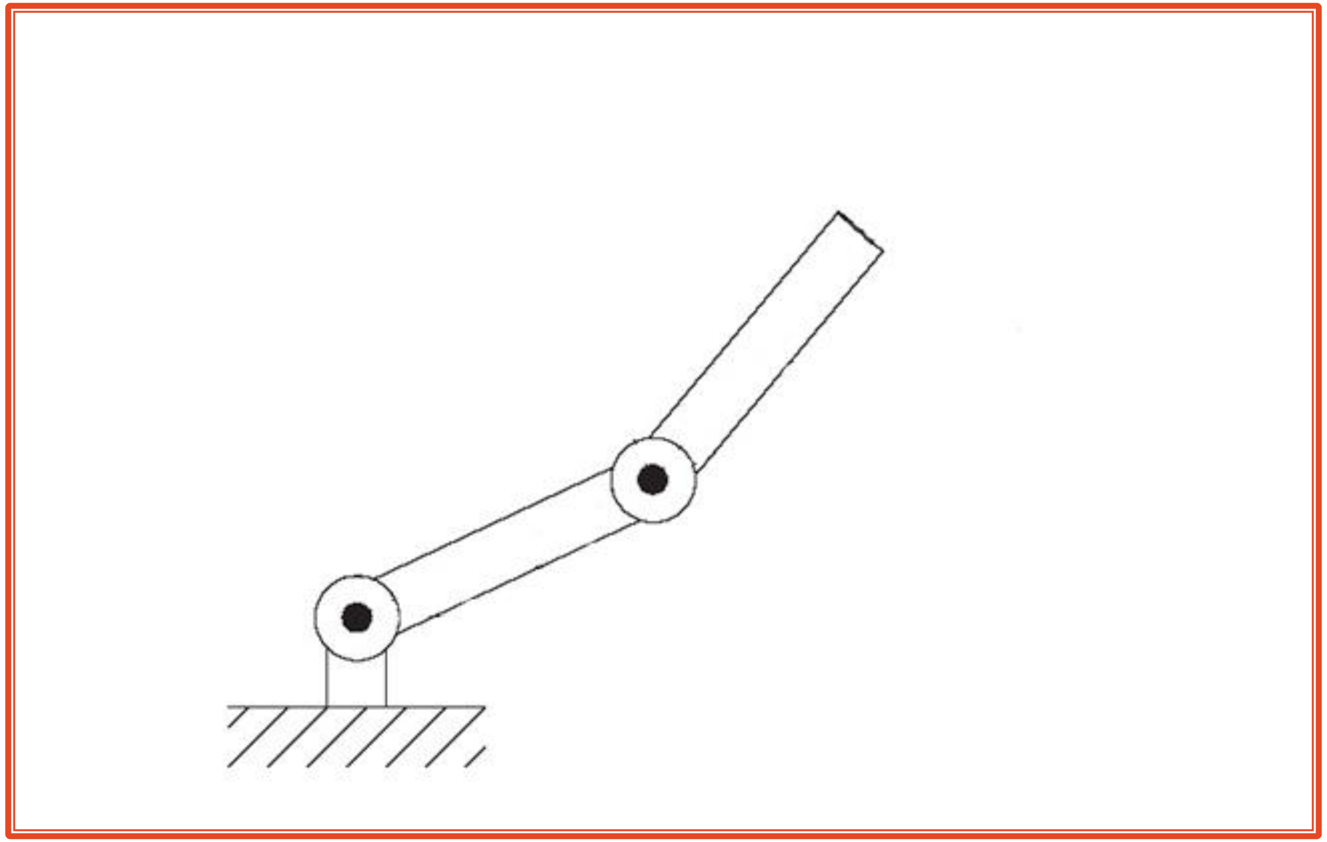
- Recall:

$$H = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{d} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

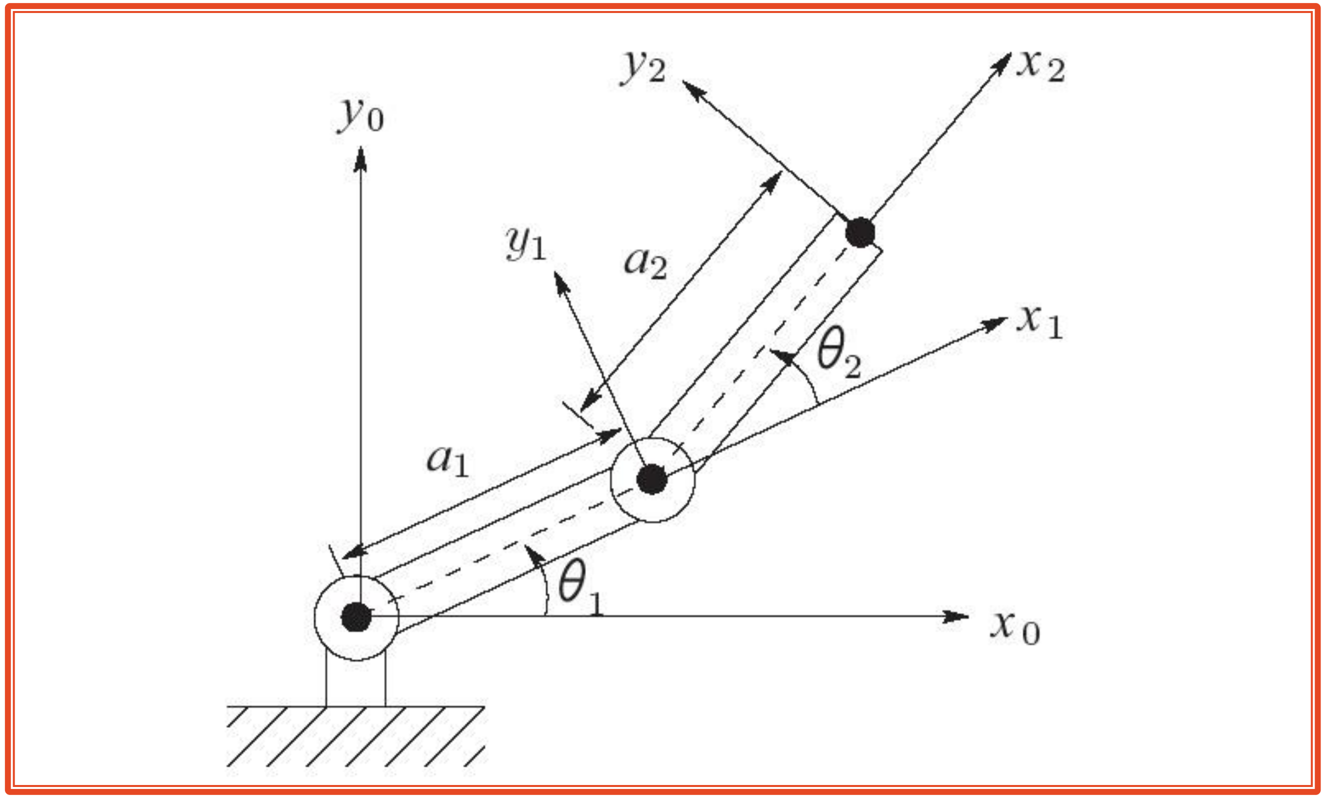
  - $\mathbf{n}$ = normal direction
  - $\boldsymbol{s}$ = sliding direction
  - $\boldsymbol{a}$ = approach direction



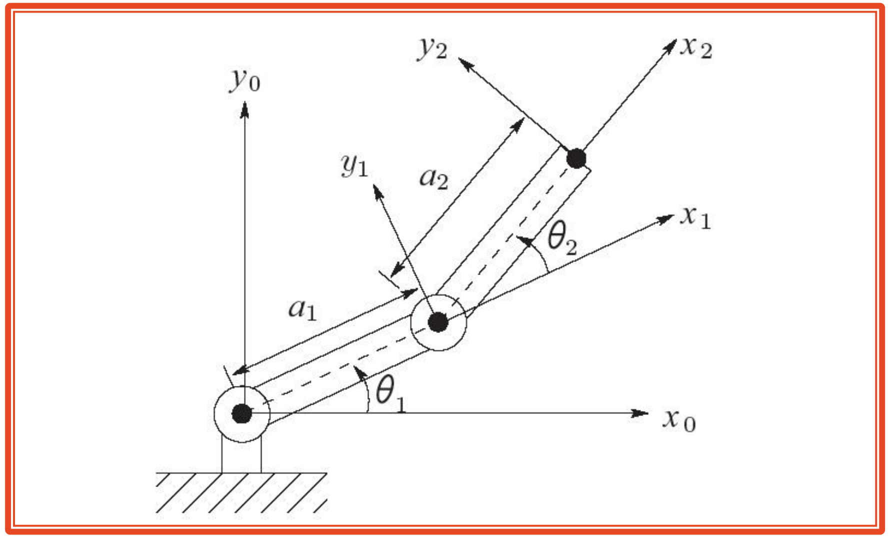
Example: A planar robot



Example: A planar robot



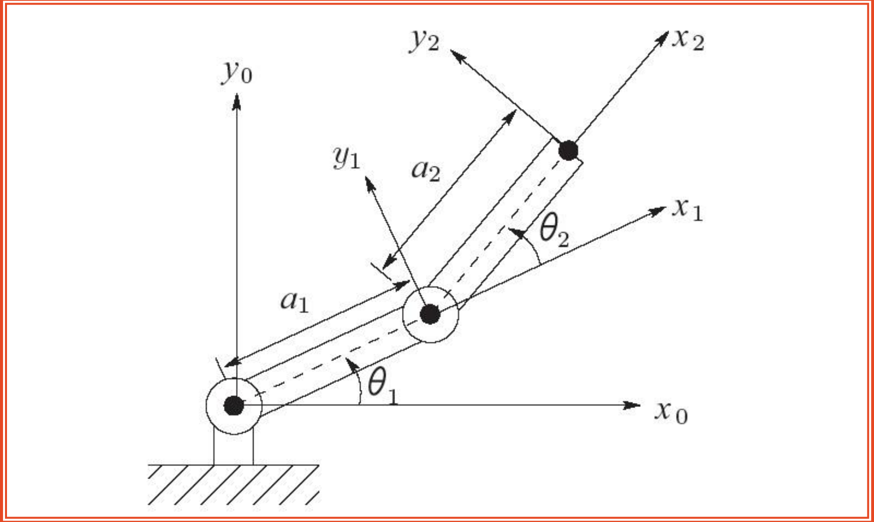
#	θ	d	a	α
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0



$$T_0^1 = A_0^1 = Rot(z,\theta_1)\cdot Trans(0,0,d_1)\cdot Trans(a_1,0,0)\cdot Rot(x,\alpha_1)$$

$$= \begin{bmatrix} c_{\theta_1} & -s_{\theta_1}c_{\alpha_1} & s_{\theta_1}s_{\alpha_1} & a_1c_{\theta_1} \\ s_{\theta_1} & c_{\theta_1}c_{\alpha_1} & -c_{\theta_1}s_{\alpha_1} & a_1s_{\theta_1} \\ 0 & s_{\alpha_1} & c_{\alpha_1} & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & a_1c_{\theta_1} \\ s_{\theta_1} & c_{\theta_1} & 0 & a_1s_{\theta_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

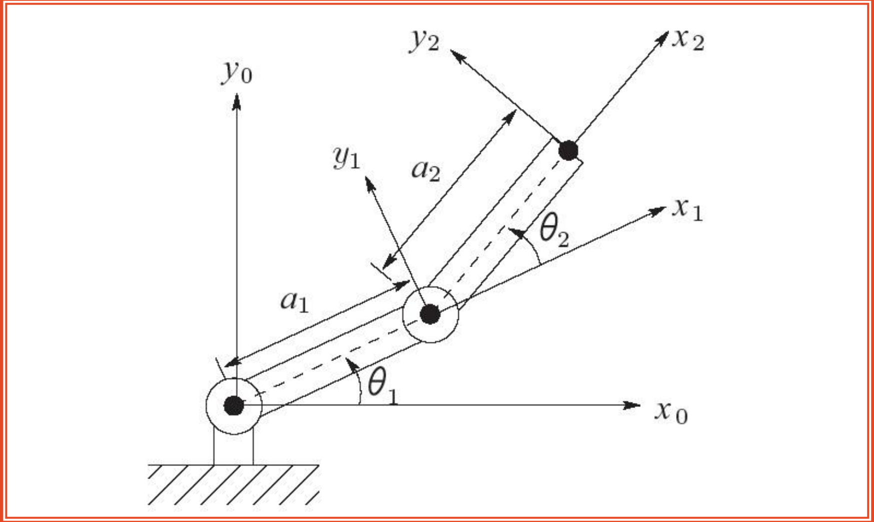
#	θ	d	a	α
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0



$$T_1^2 = A_1^2 = Rot(z,\theta_2)\cdot Trans(0,0,d_2)\cdot Trans(a_2,0,0)\cdot Rot(x,\alpha_2)$$

$$= \begin{bmatrix} c_{\theta_2} & -s_{\theta_2}c_{\alpha_2} & s_{\theta_2}s_{\alpha_2} & a_2c_{\theta_2} \\ s_{\theta_2} & c_{\theta_2}c_{\alpha_2} & -c_{\theta_2}s_{\alpha_2} & a_2s_{\theta_2} \\ 0 & s_{\alpha_2} & c_{\alpha_2} & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 & a_2c_{\theta_2} \\ s_{\theta_2} & c_{\theta_2} & 0 & a_2s_{\theta_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

#	θ	d	a	α
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0

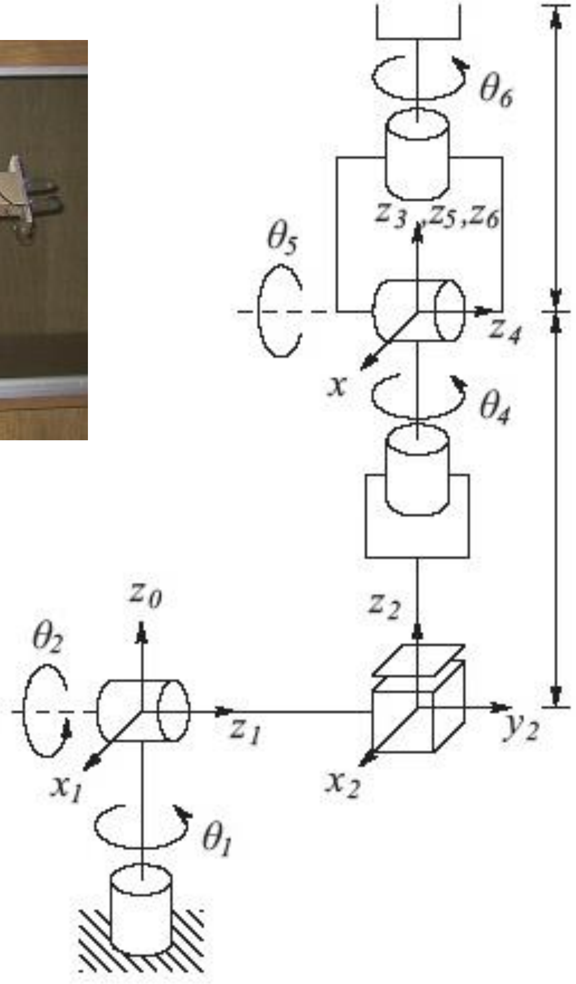


$$A_0^1 = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & a_1 c_{\theta_1} \\ s_{\theta_1} & c_{\theta_1} & 0 & a_1 s_{\theta_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1^2 = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 & a_2 c_{\theta_2} \\ s_{\theta_2} & c_{\theta_2} & 0 & a_2 s_{\theta_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

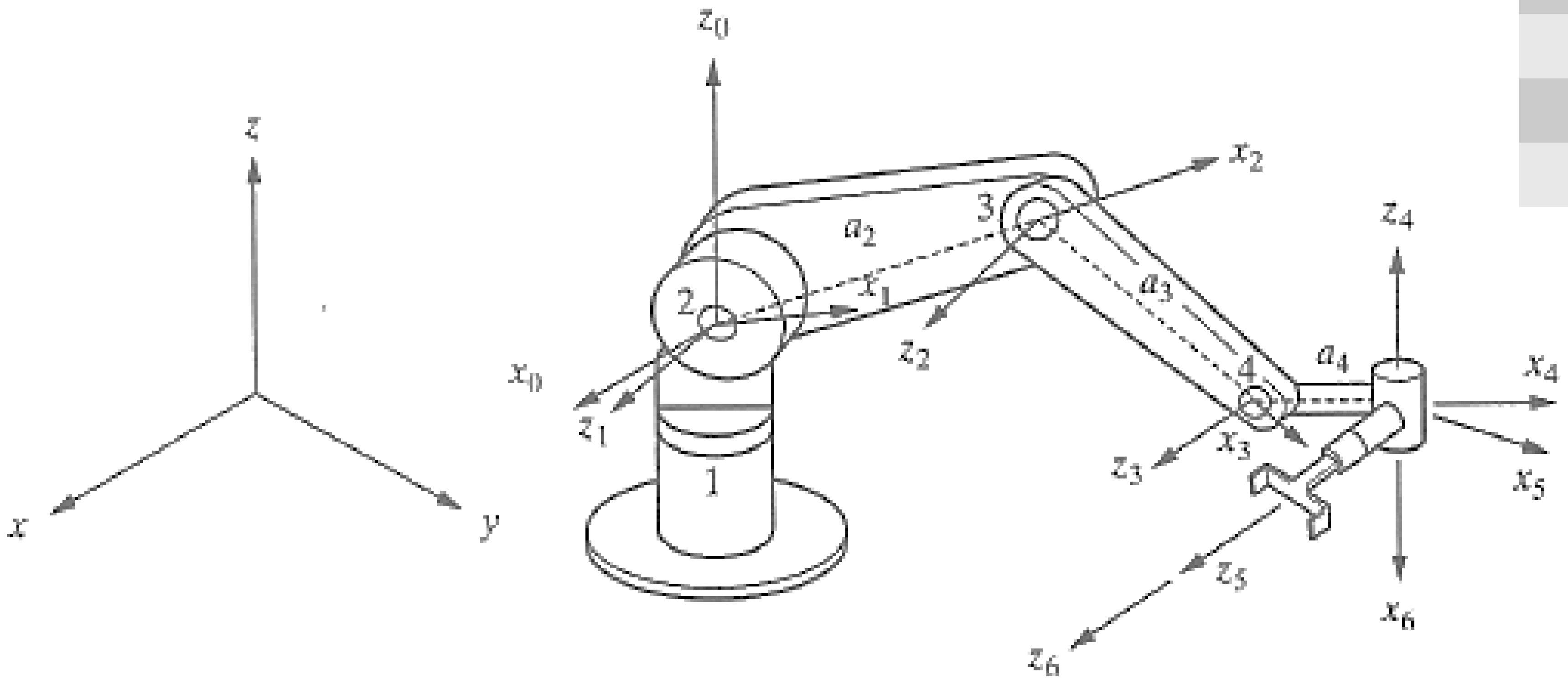
$$T_0^1 = A_0^1$$

$$T_0^2 = A_0^1 A_1^2 = \begin{bmatrix} c_{\theta_1} c_{\theta_2} - s_{\theta_1} s_{\theta_2} & -c_{\theta_1} s_{\theta_2} - s_{\theta_1} c_{\theta_2} & 0 & a_1 c_{\theta_1} + a_2 \left(c_{\theta_1} c_{\theta_2} - s_{\theta_1} s_{\theta_2} \right) \\ s_{\theta_1} c_{\theta_2} + c_{\theta_1} s_{\theta_2} & c_{\theta_1} c_{\theta_2} - s_{\theta_1} s_{\theta_2} & 0 & a_1 s_{\theta_1} + a_2 \left(s_{\theta_1} c_{\theta_2} + c_{\theta_1} s_{\theta_2} \right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

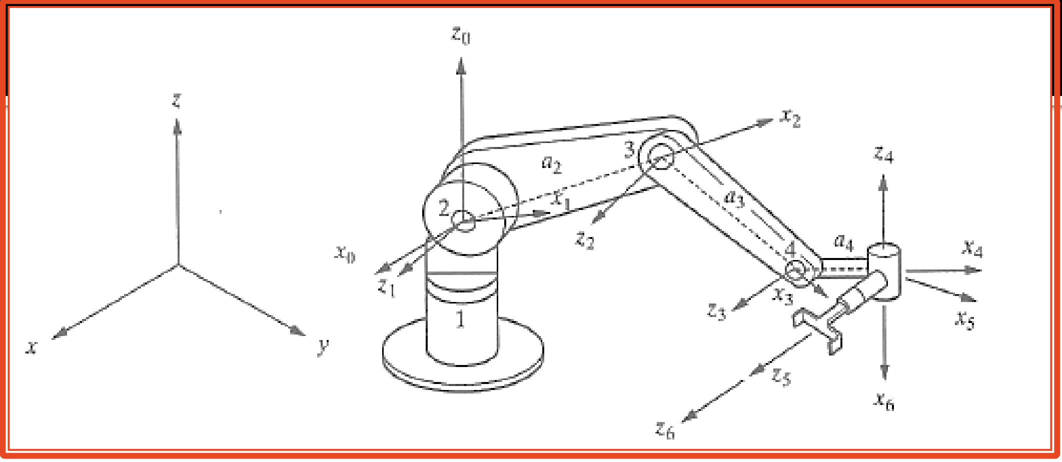


Link	d_i	a_i	α_i	θ_i
1	0	0	−90	$\theta_1^\star$
2	d_2	0	+90	$\theta_2^\star$
3	$d_3^\star$	0	0	0
4	0	0	−90	$\theta_4^\star$
5	0	0	+90	$\theta_5^\star$
6	d_6	0	0	$\theta_6^\star$

Example: A six-DOF articulate robot



#	θ	d	a	α
1	θ_1	0	0	90
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0
4	θ_4	0	a_4	-90
5	θ_5	0	0	90
6	θ_6	0	0	0



$$T_0^1 = A_0^1 A_1^2 A_2^3 A_3^4 A_4^5 A_5^6$$

$$= \begin{bmatrix} c_1\left(c_{234}c_5c_6 - s_{234}s_6\right) & c_1\left(-c_{234}c_5c_6 - s_{234}c_6\right) & c_1\left(c_{234}s_5\right) & c_1\left(c_{234}a_4 + c_{23}a_3 + c_2a_2\right) \\ -s_1s_5c_6 & +s_1s_5s_6 & +s_1c_5 & \\ s_1\left(c_{234}c_5c_6 - s_{234}s_6\right) & s_1\left(-c_{234}c_5c_6 - s_{234}c_6\right) & s_1\left(c_{234}s_5\right) & s_1\left(c_{234}a_4 + c_{23}a_3 + c_2a_2\right) \\ +c_1s_5s_6 & -c_1s_5s_6 & -c_1c_5 & \\ s_{234}c_5c_6 + c_{234}s_6 & -s_{234}c_5c_6 + c_{234}c_6 & s_{234}s_5 & s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Singularities

- There are 3 common singularities with serial robotics systems
  - Wrist alignment – joint 4 and 6 – collinear axis
  - Elbow singularity - Out-of-reach
  - Alignment singularity – wrist is as close to joint 1 as it can get

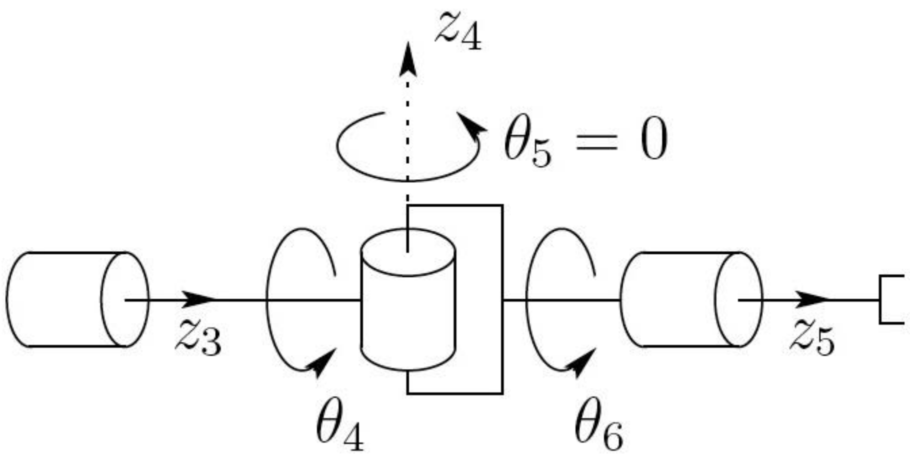
Degeneracy

- Degeneracy = Robot looses 1 DOF
  - Physical Limits
  - 2 similar joints become collinear
  - Determinant of position matrix = zero

- Reduced dexterity
  - Impossible to orient end effecter at a desired orientation, at the limits of robots workspace.

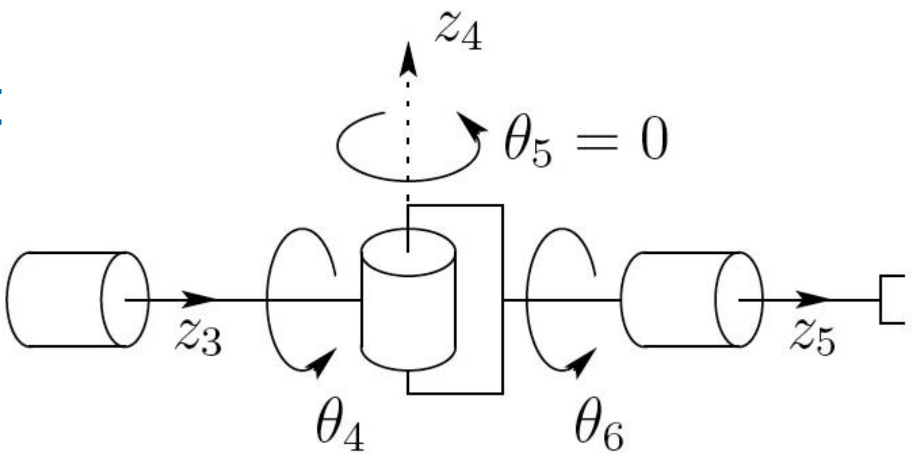
Example: Spherical wrist singularity

- A spherical wrist
  - A singular configuration when the vectors z3 and z5 are linearly dependent.
  - The axes z3 and z5 are collinear, which happens when θ5 = 0 or π.

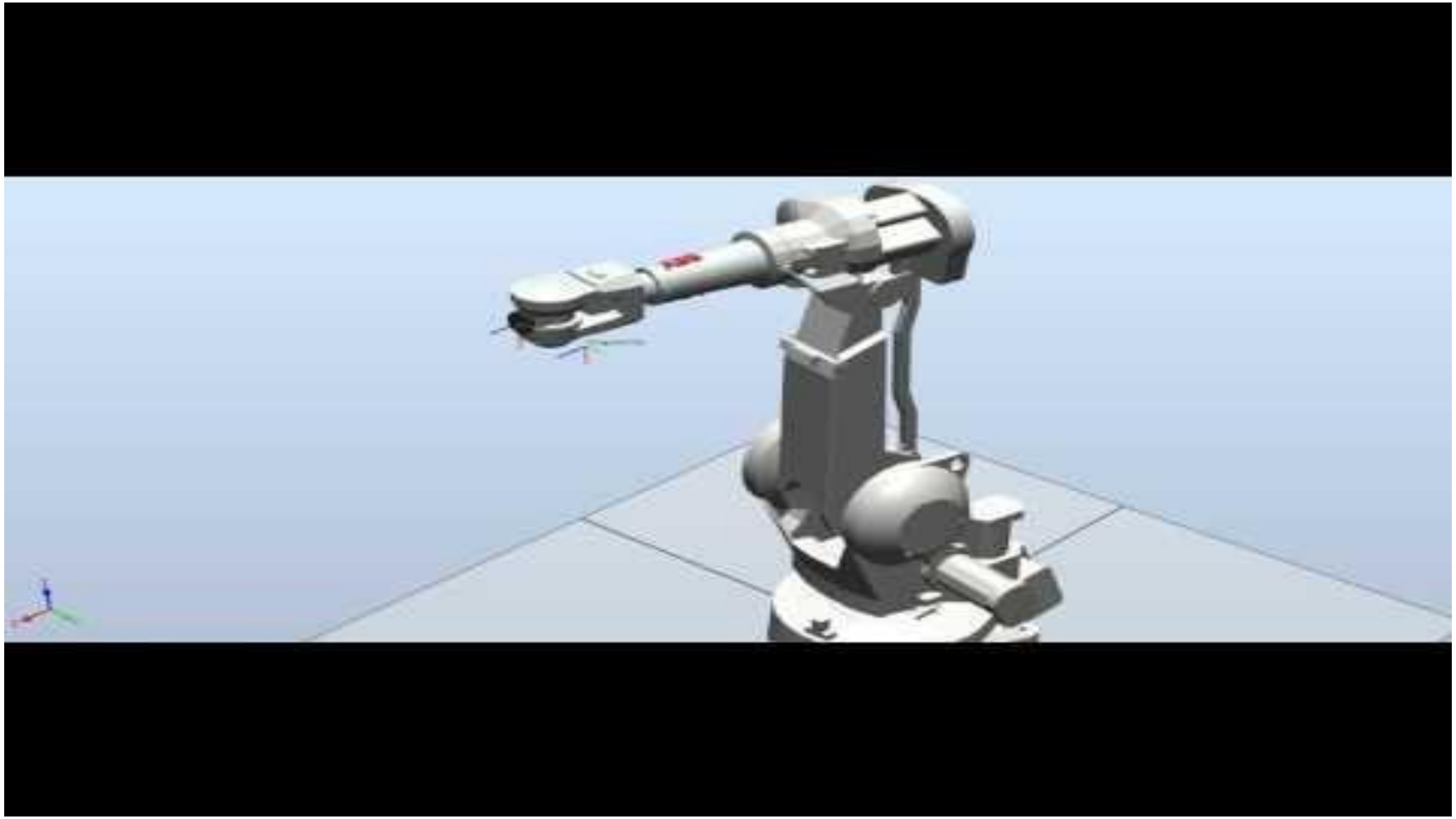


Example: Spherical wrist singularity

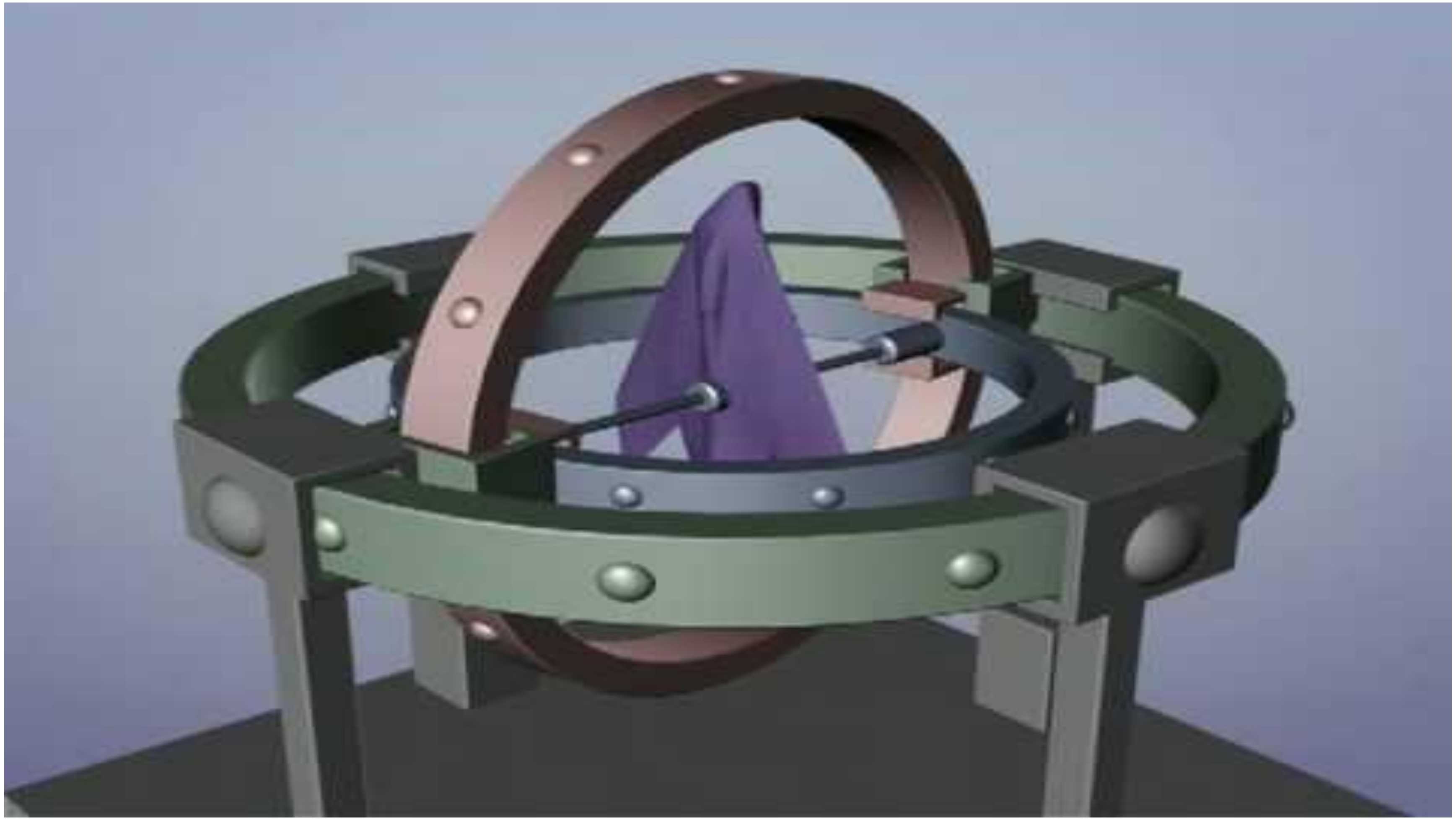
- Unavoidable singularity for sphere wrist, unless …
  - The wrist is designed in such a way as t not permit this alignment.
- Not limited to a spherical wrist
  - If any two revolute joint axes become collinear a singularity results.



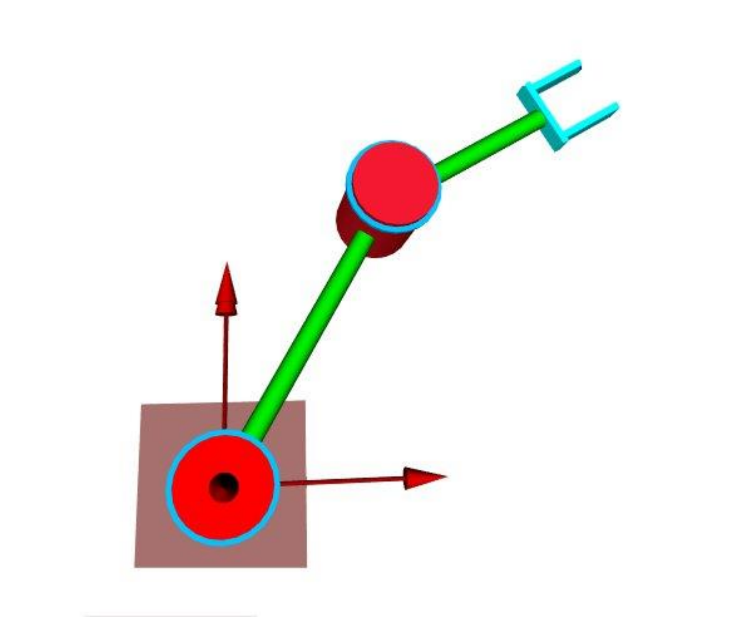
Wrist singularity



Gimbal Lock

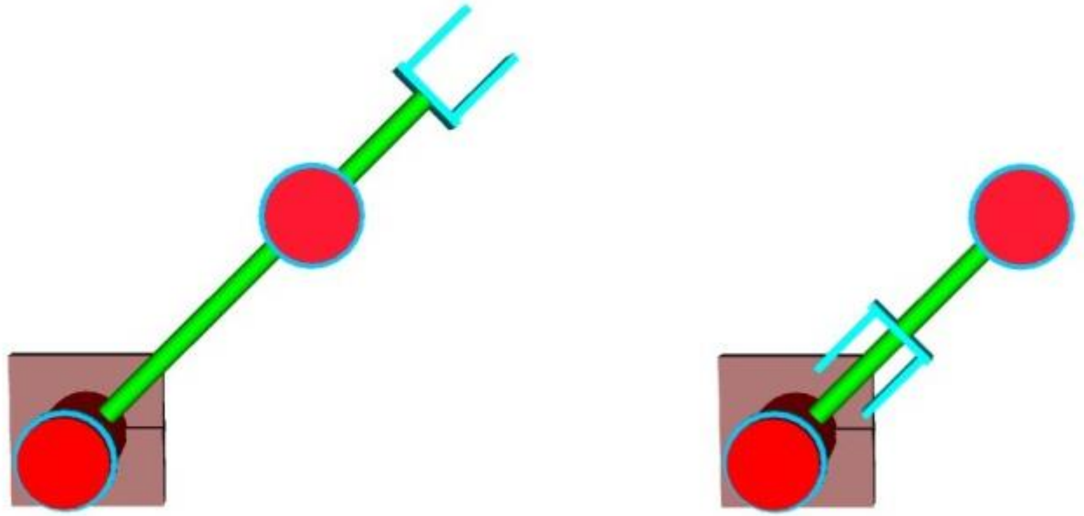


2D Elbow Singularities



- The robot arm has two joints
- The joint space has 2 dimensions
- Theoretically, any position within the robot workspace is reachable by the end effector. However …
  - A singularity reduces the mobility of the robot.
  - This will occur in two configurations – what are they?

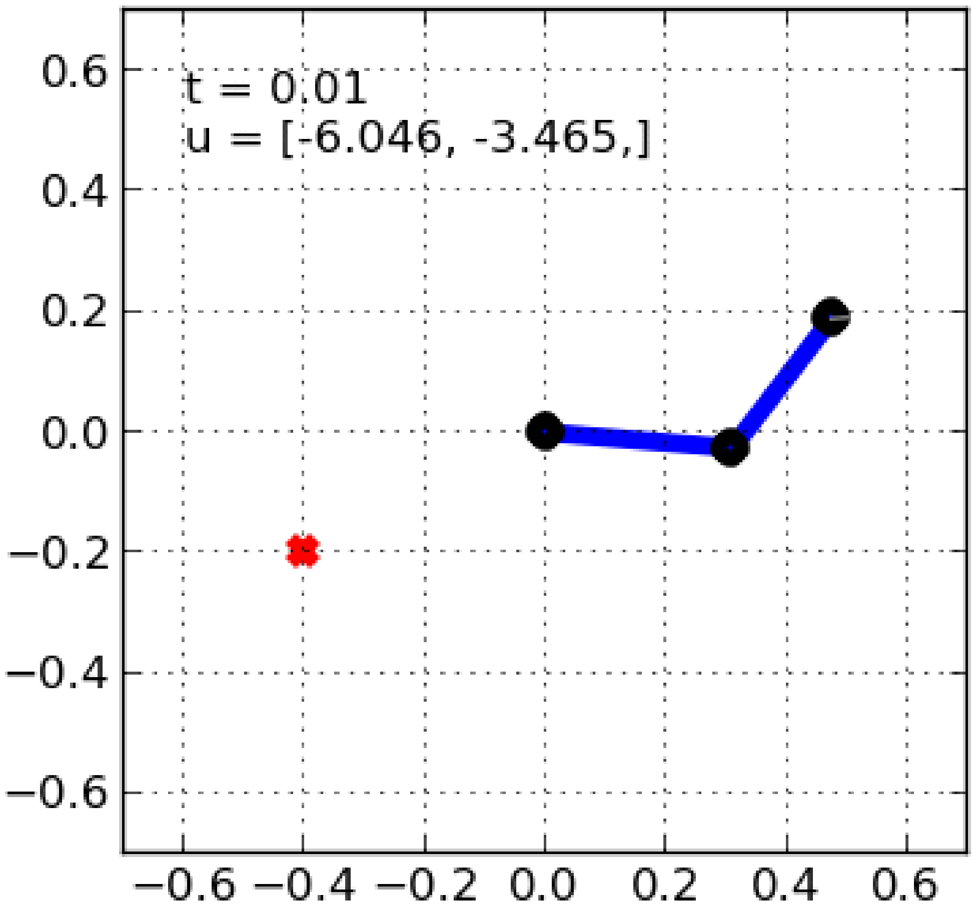
2D Elbow Singularities



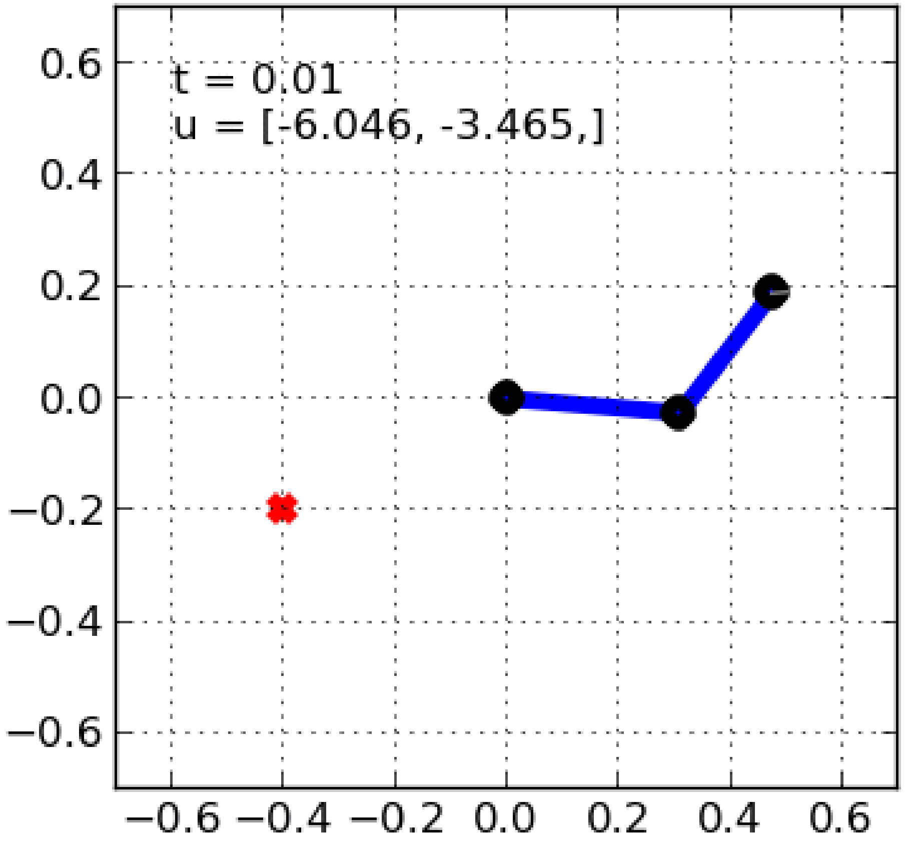
$\theta_2 = 0$
Arm fully extended

$\theta_2 = \pi$
Arm fully retracted

2D Elbow singularity

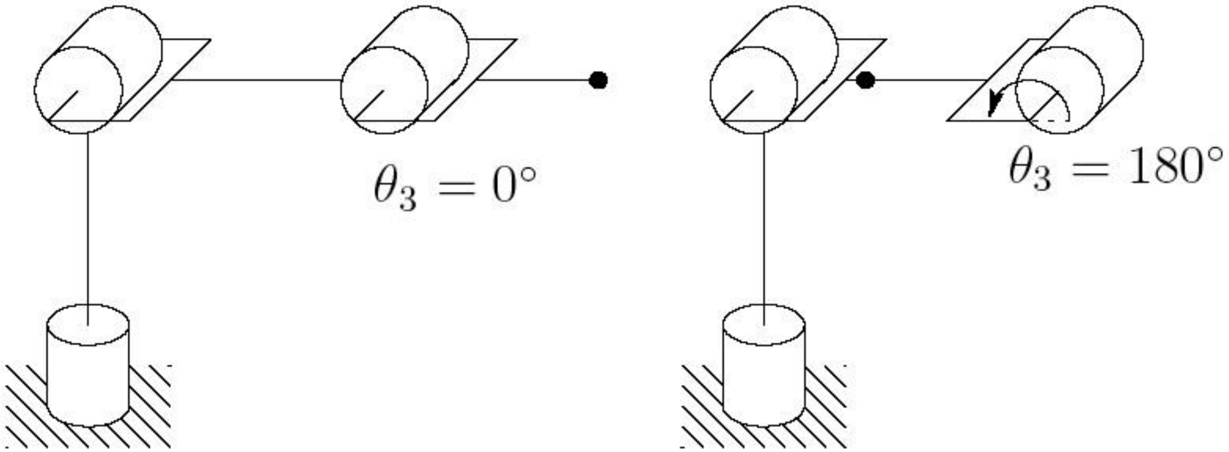


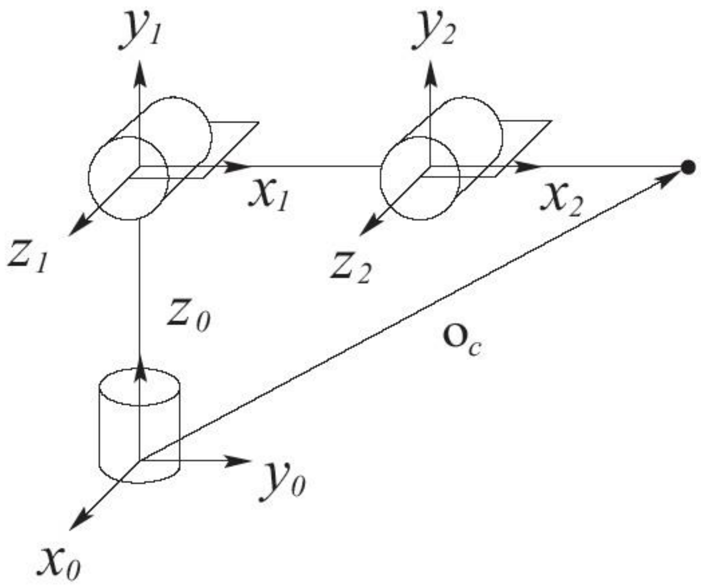
2D Elbow Singularities



3D example

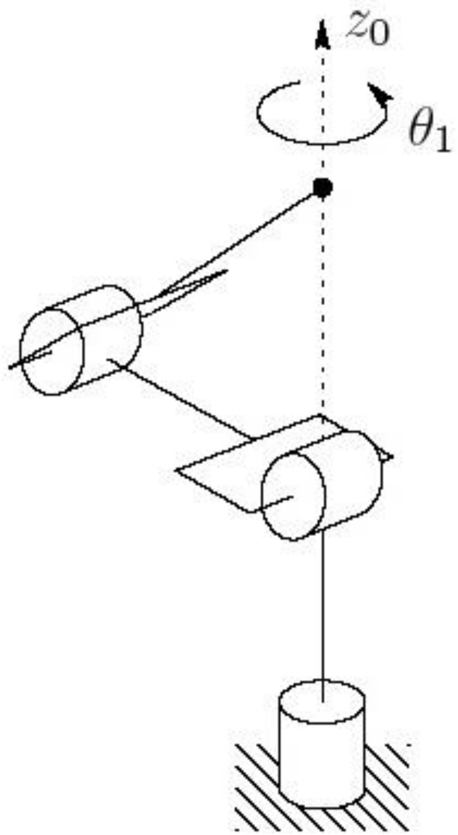
- Singularity due to aligned links
  - Like the 2D case we just saw, there are two singularities due to the parallel Z1 and Z2 axes.





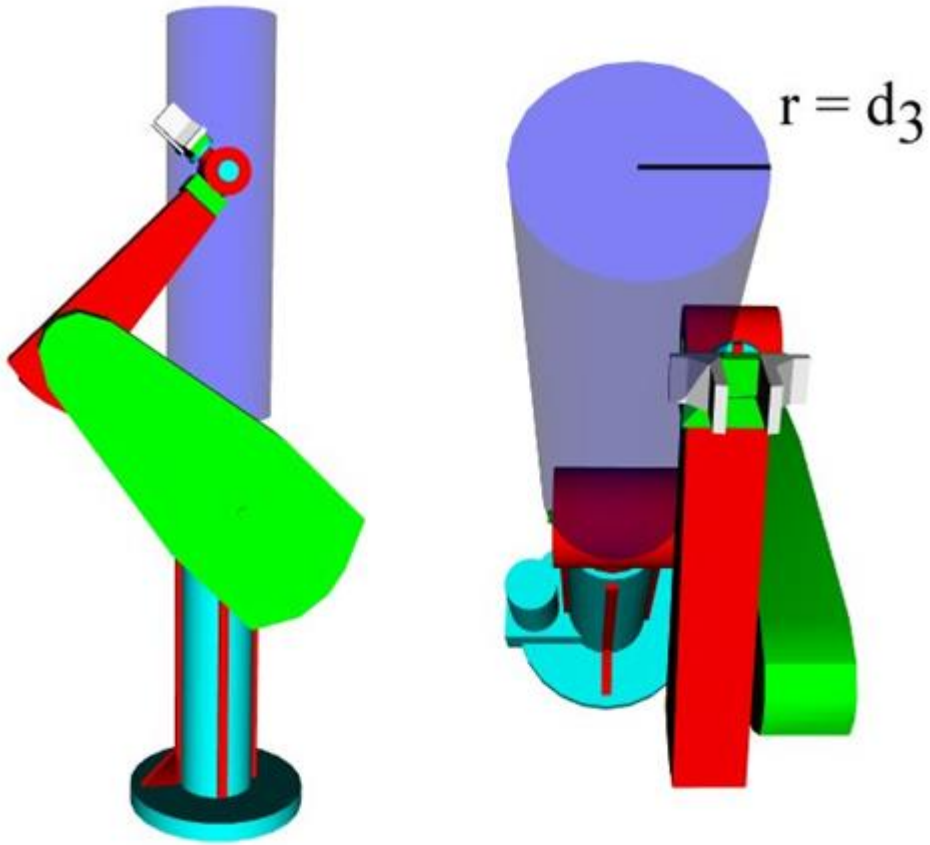
3D example

- Singularity due to aligned rotational axes
  - If the wrist center intersects with the axis of the base rotation, z_0, then there are an infinite number of solutions to the inverse kinematic equations.
  - In other words, any value of θ1 will produce the same wrist position. We have again lost a degree of freedom…



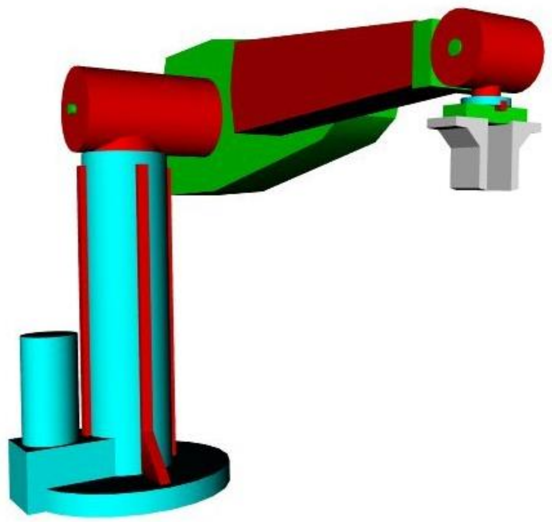
3D example

- Singularity due to reach limit
  - There are workspace volumes (shown in purple) where the end of arm tooling cannot reach.



3D example

- Singularity due to self collision
  - There are also configurations where the arm will collide with itself (another form of singularity).



Singularities of the ABB robot

End