Transformation

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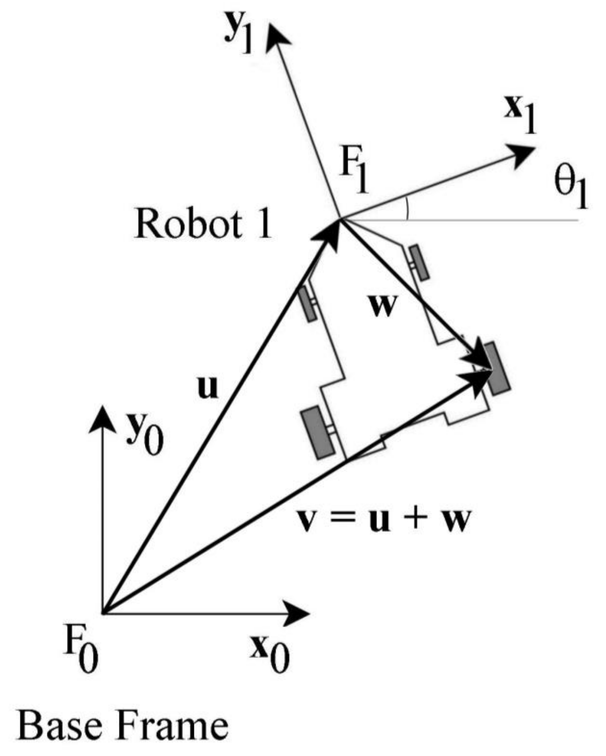
Quiz (10 pts)

- Given that
  - The center of the right rear wheel is at planar coordinates (0.24, -0.53) w.r.t. frame F_1
  - $a_1 = 7$, $b_1 = 3$, and $\theta_1 = 26°$
  - (4 pts) Use homogeneous point vector to express the position of this wheel in Frame F_1
  - (6 pts) Use homogeneous transformation matrix to express this wheel w.r.t. Frame F_0

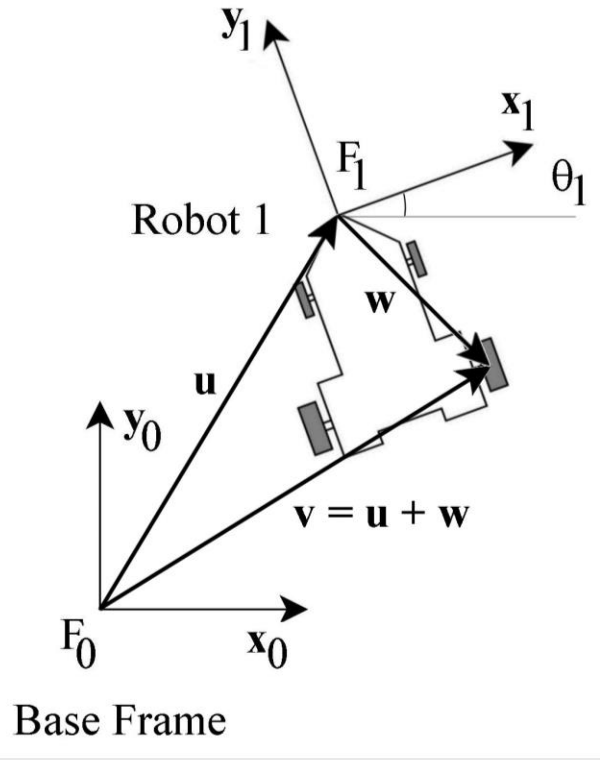


Alternative solution?

- A more efficient way to solve the problem is to use the combined matrix $\mathbf{P}$:

$$\mathbf{v} = \mathbf{P}\mathbf{w} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & a_1 \\ \sin(\theta_1) & \cos(\theta_1) & b_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + x_w \cos(\theta_1) - y_w \sin(\theta_1) \\ b_1 + x_w \sin(\theta_1) + y_w \cos(\theta_1) \\ 1 \end{bmatrix}$$



Transformation

Why need transformation?

- We want to control the end-effector of a robot to …
  - Move to the desired pose (position and orientation)
  - Move along a pre-planned path, i.e., a sequence of robot poses

How to represent the robot end-effector poses mathematically?

Overview

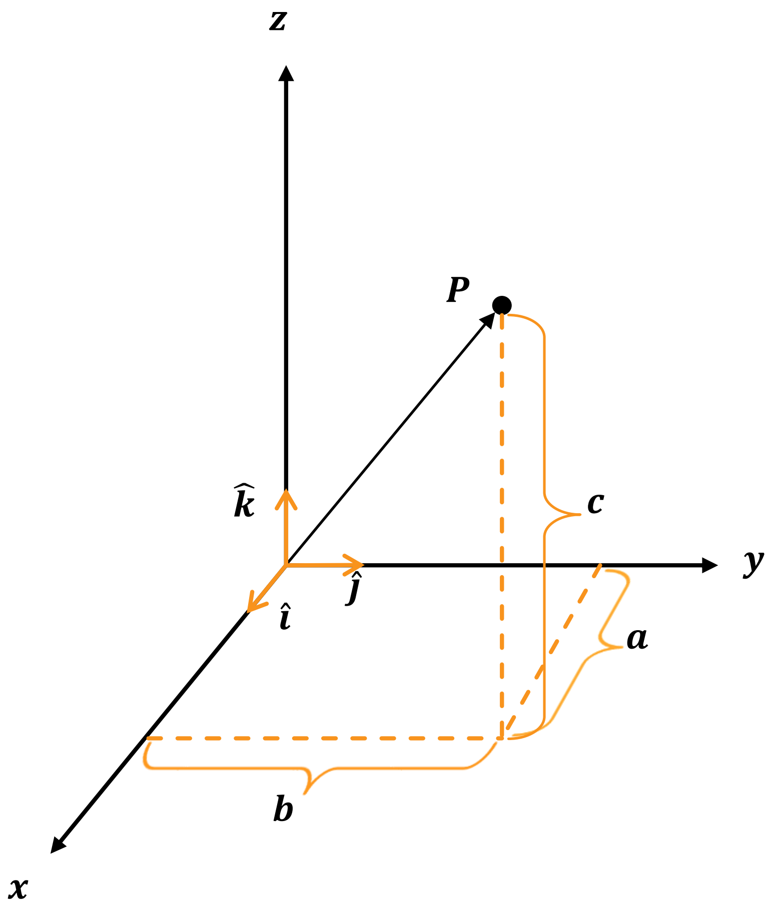
- Reference Frames and Coordinate Systems
- Representing a Point and Vector in Space
  - Representing Rotations
  - Rotations in 2D
  - Characteristics of Rotation Matrices
  - Rotations in 3D
  - Rotational Transformations
  - Rigid Motion: Rotation and Translation
- Homogeneous Transformations

Representing a Point and Vector in Space

- Normal representation of a point

- Representation using unit vector

$$P = a\hat{i} + b\hat{j} + c\hat{k}$$

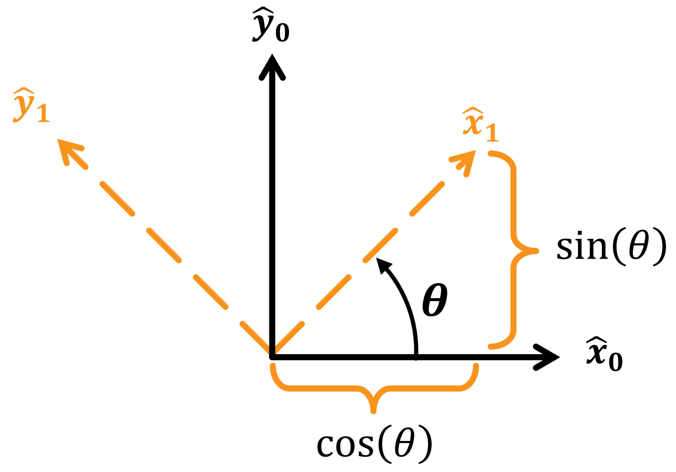


Representing Rotations in 2D

- What we want to know what is $R_0^1 = \left[{}^0\hat{x}_1 \mid {}^0\hat{y}_1 \right]$
  - where ${}^0\hat{x}_1$ and ${}^0\hat{y}_1$ are the coordinates in frame 0 of the unit vectors and , respectively. A matrix in this form is called a rotation matrix.

$$ {}^0\hat{x}_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad {}^0\hat{y}_1 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} $$

$$ R_0^1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} $$



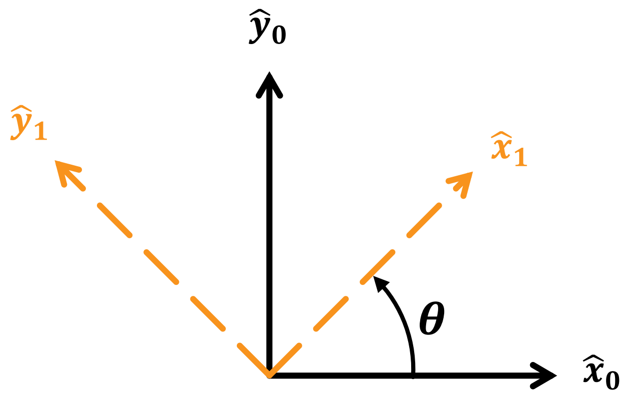
Representing Rotations in 2D

- Alternatively, we can derive the 2D transformation matrix by
  - Project Frame 1 axes onto Frame 0 axes:

$$ {}^{0}\hat{x}_1 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 \end{bmatrix} \qquad {}^{0}\hat{y}_1 = \begin{bmatrix} \hat{y}_1 \cdot \hat{x}_0 \\ \hat{y}_1 \cdot \hat{y}_0 \end{bmatrix} $$

  - Combine into a single matrix

$$ R_0^1 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} $$



Inverse Rotation

- What is the rotation of Frame 0 w.r.t. Frame 1?

$$R_1^0 = \begin{bmatrix} \hat{x}_0 \cdot \hat{x}_1 & \hat{y}_0 \cdot \hat{x}_1 \\ \hat{x}_0 \cdot \hat{y}_1 & \hat{y}_0 \cdot \hat{y}_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

- Compare to
-

$$R_0^1 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\Rightarrow \quad R_1^0 = \left[R_0^1 \right]^T$$

Characteristics of Rotation Matrices

- Special orthogonal group

$$R \in SO(n)$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

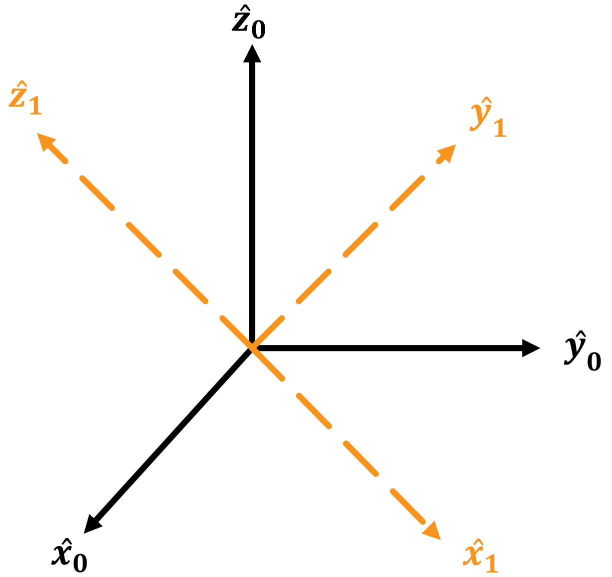
- For any $R \in SO(n)$, the following properties hold
  - $R^T = R^{-1} \in SO(n)$
  - Columns (and rows) of *R* are **mutually orthogonal** ⇒ $\mathbf{u} \cdot \mathbf{v} = 0$
  - Each column (and each row) of *R* is a unit vector
  - $\det(R) = 1$ (-1 for left handed coordinate systems)

Representing Rotations in 3D

- Project Frame 1 axes onto Frame 0 axes:

$$R_0^1 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot \hat{z}_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix}$$

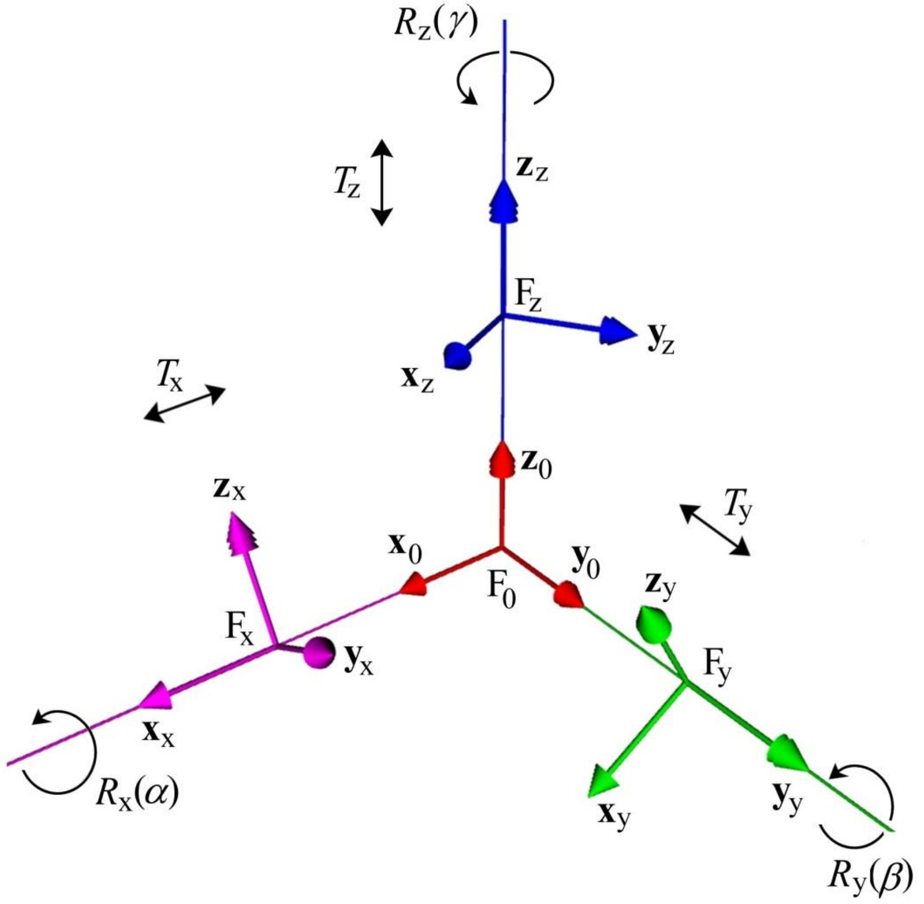
$$R_0^1 \in SO(3)$$



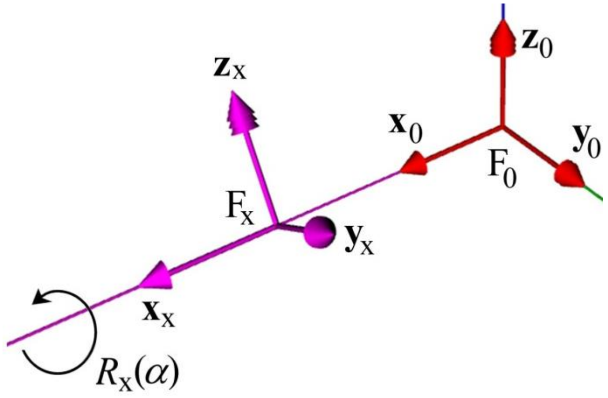
Basic (Canonical) Rotations

$$R_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \quad R_{y,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad R_{z,\gamma} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Representing Rotations in 3D



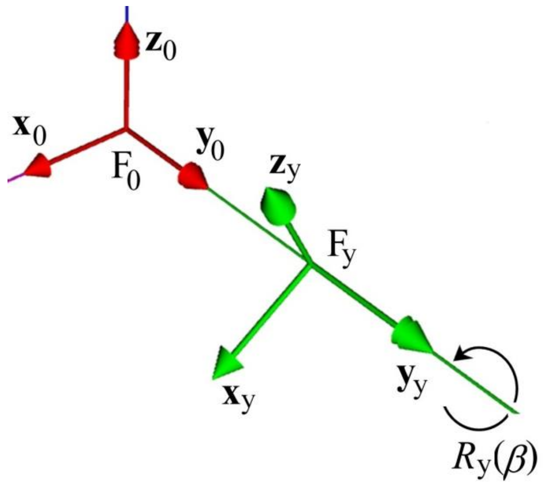
Representing Rotations in 3D



$$\mathbf{R}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

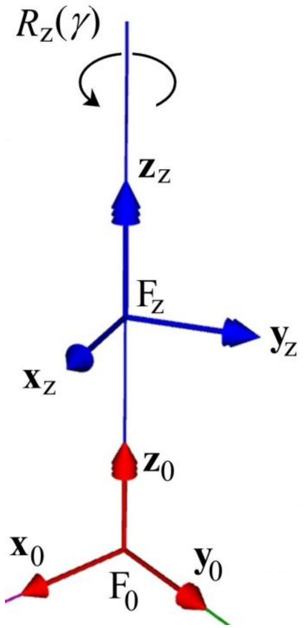
$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix}$$

Representing Rotations in 3D



$$\mathbf{R}_y(\beta) = \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix}$$

Representing Rotations in 3D



$$\mathbf{R}_z(\gamma) = \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

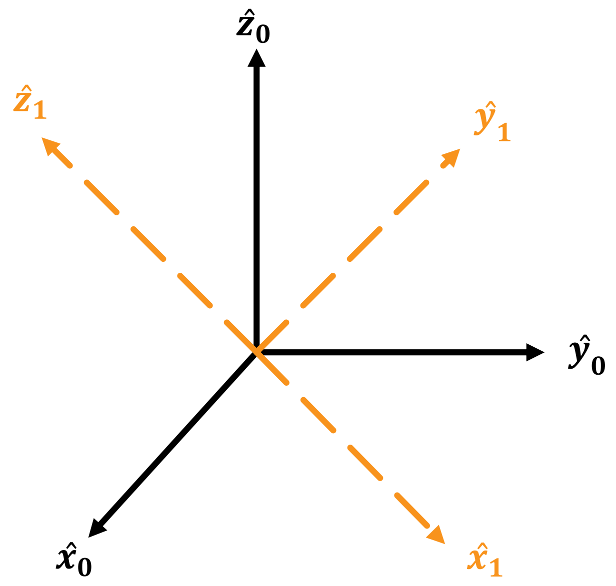
Representing Rotations in 3D

- Canonical Rotations

$$R_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_{y,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_{z,\gamma} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

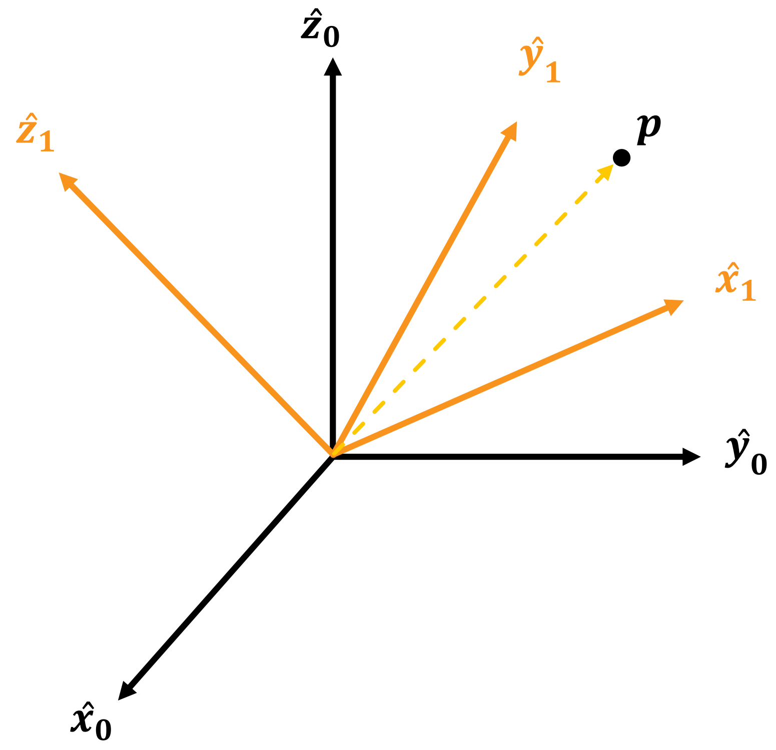


Representing Rotations in 3D

- Project the point $\boldsymbol{p}_1$ onto Frame 0:

$$p_0 = \begin{bmatrix} p_1 \cdot \hat{x}_0 \\ p_1 \cdot \hat{y}_0 \\ p_1 \cdot \hat{z}_0 \end{bmatrix} = \begin{bmatrix} (u\hat{x}_1 + v\hat{y}_1 + w\hat{z}_1) \cdot \hat{x}_0 \\ (u\hat{x}_1 + v\hat{y}_1 + w\hat{z}_1) \cdot \hat{y}_0 \\ (u\hat{x}_1 + v\hat{y}_1 + w\hat{z}_1) \cdot \hat{z}_0 \end{bmatrix}$$

$$= \begin{bmatrix} u\hat{x}_1 \cdot \hat{x}_0 & v\hat{y}_1 \cdot \hat{x}_0 & w\hat{z}_1 \cdot \hat{x}_0 \\ u\hat{x}_1 \cdot \hat{y}_0 & v\hat{y}_1 \cdot \hat{y}_0 & w\hat{z}_1 \cdot \hat{y}_0 \\ u\hat{x}_1 \cdot \hat{z}_0 & v\hat{y}_1 \cdot \hat{z}_0 & w\hat{z}_1 \cdot \hat{z}_0 \end{bmatrix}$$



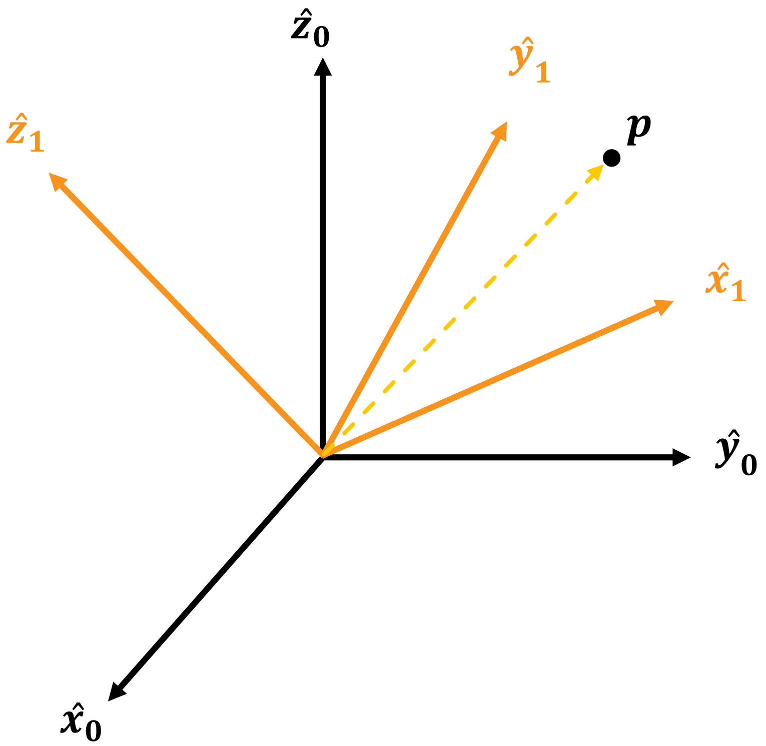
Representing Rotations in 3D

- Project the point $\boldsymbol{p}_1$ onto Frame 0:

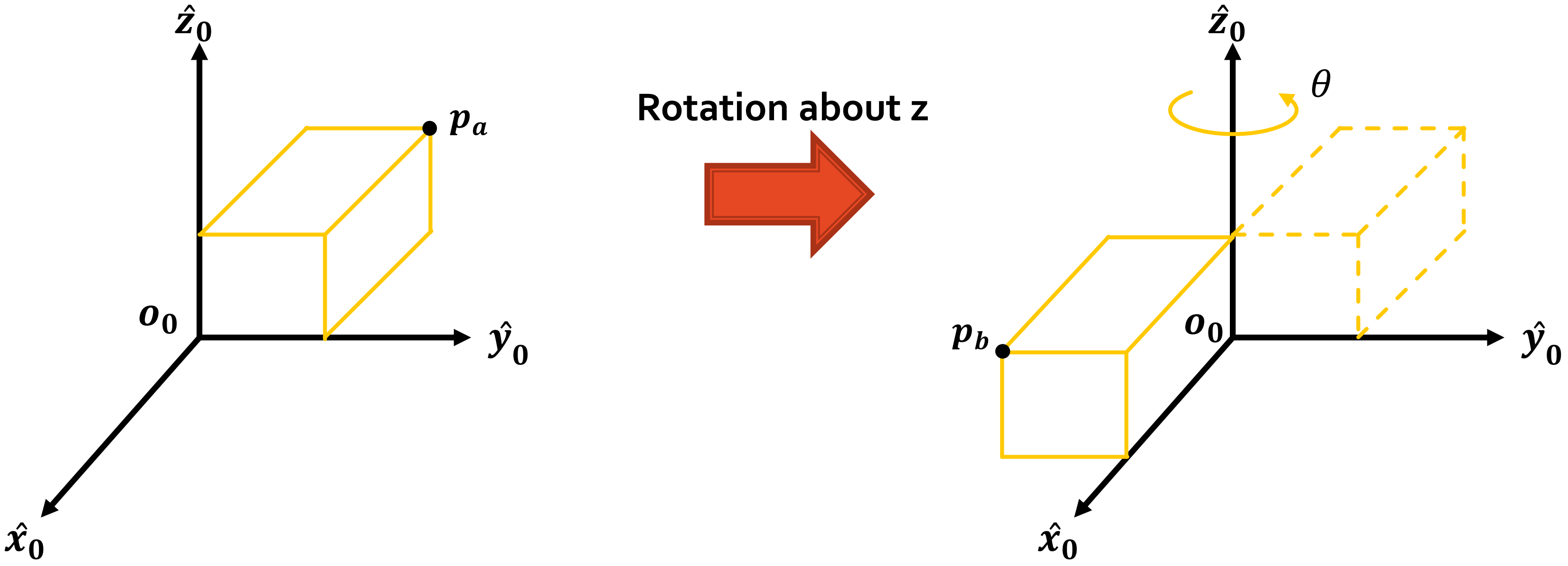
$$p_0 = \begin{bmatrix} u\hat{x}_1 \cdot \hat{x}_0 & v\hat{y}_1 \cdot \hat{x}_0 & w\hat{z}_1 \cdot \hat{x}_0 \\ u\hat{x}_1 \cdot \hat{y}_0 & v\hat{y}_1 \cdot \hat{y}_0 & w\hat{z}_1 \cdot \hat{y}_0 \\ u\hat{x}_1 \cdot \hat{z}_0 & v\hat{y}_1 \cdot \hat{z}_0 & w\hat{z}_1 \cdot \hat{z}_0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot \hat{z}_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix}}_{\text{Simply } R_0^1} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{p_1}$$

Simply R_0^1 and p_1



Rotational Transformations



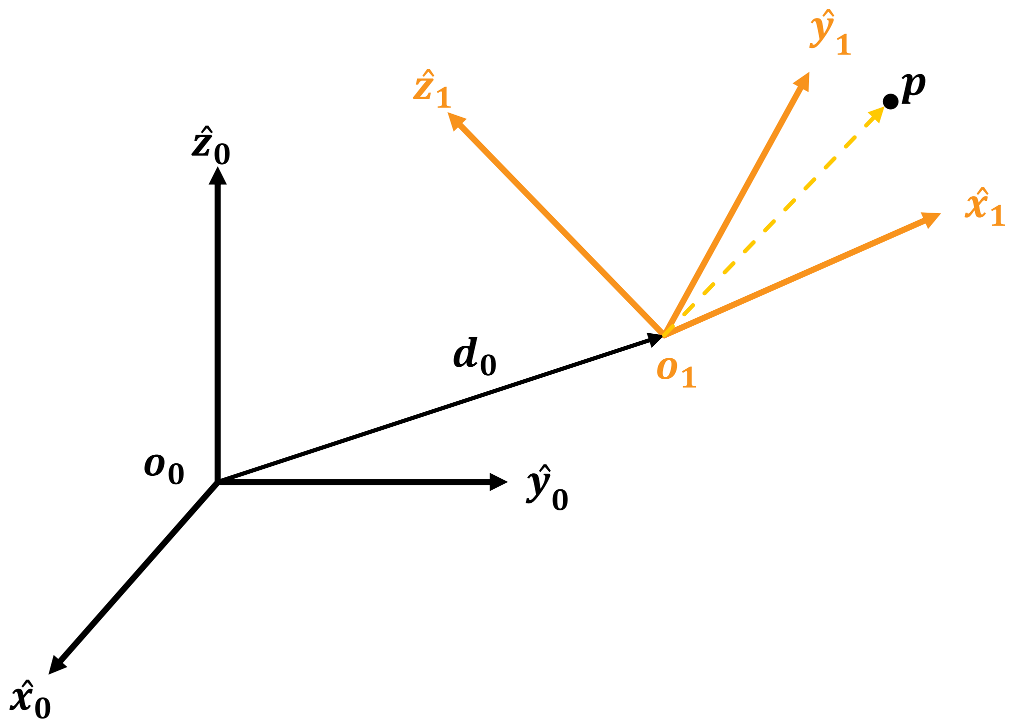
- Rotation simply represented as:

$$R_0^1 = R_{z,\pi} \Rightarrow p_{0,b} = R_{z,\pi} p_{0,a}$$

Rigid Motions: Rotation and Translation

- Given a point ***p*** in Frame 1, we can express the point in Frame 0 with:

$$p_0 = R_0^1 p_1 + d_0$$



Rigid Motions: Rotation and Translation

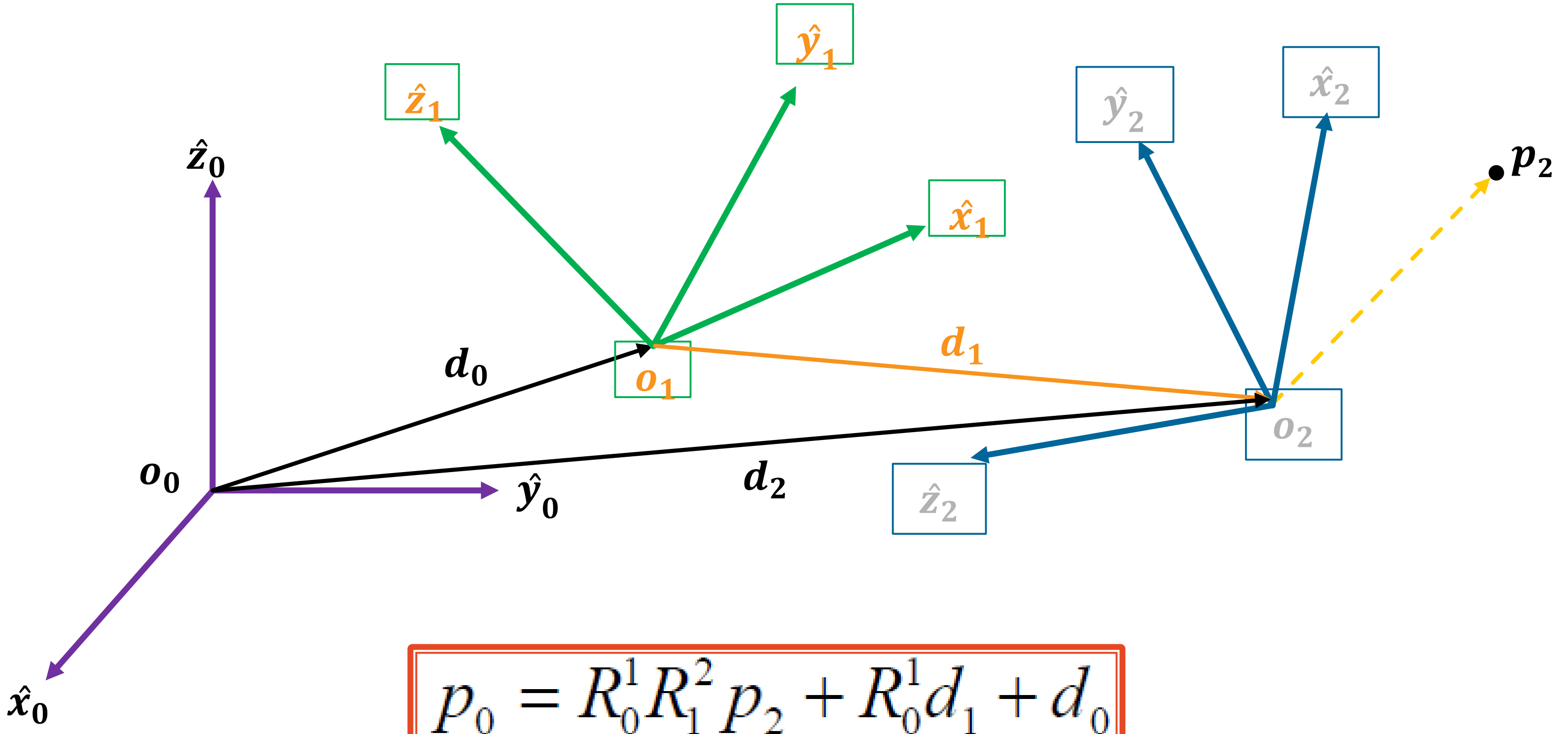
- P2 is the position of Point P defined w.r.t. Frame 2

- P1 is P2 in Frame 1

$$p_1 = R_1^2 p_2 + d_1$$

- P0 is P1 in Frame 0

$$p_0 = R_0^1 p_1 + d_0$$



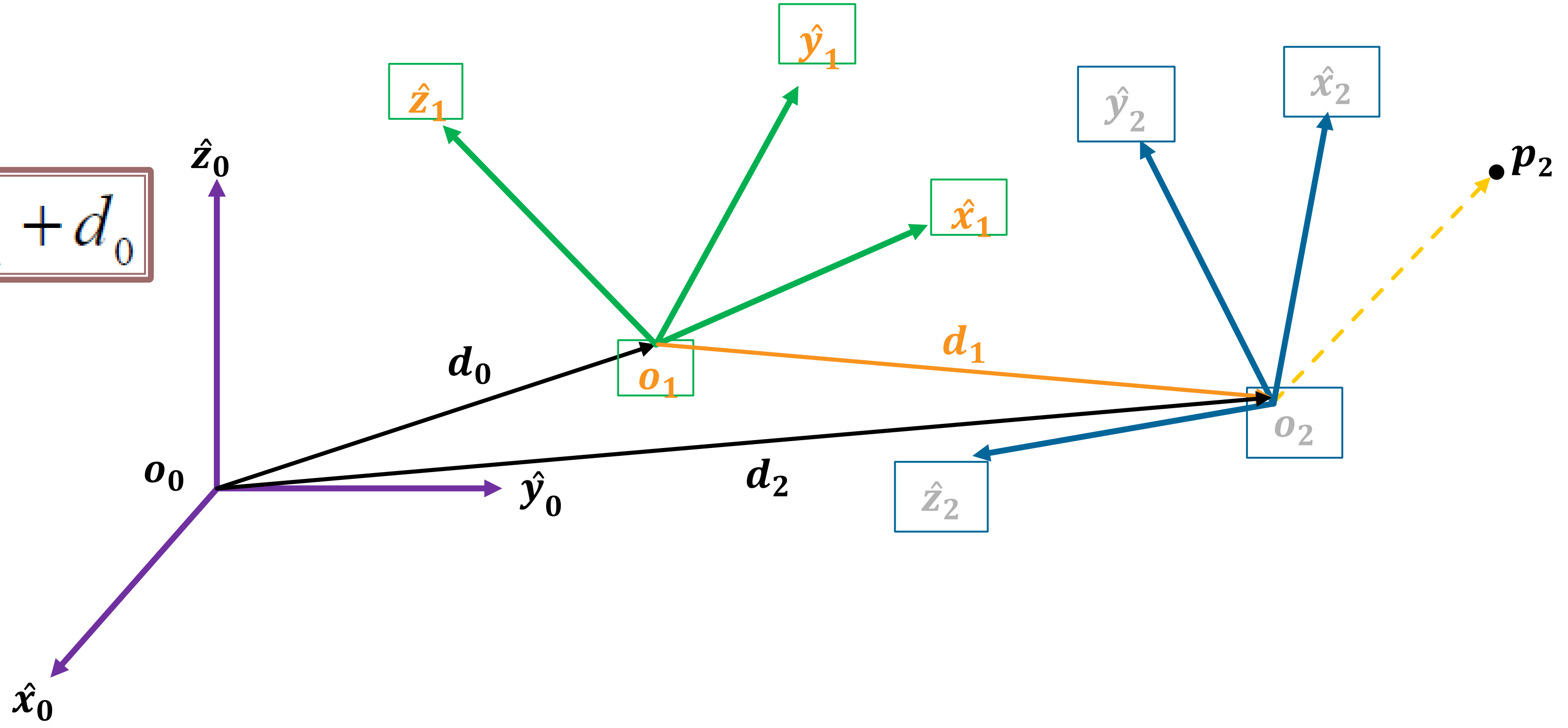
$$p_0 = R_0^1 R_1^2 p_2 + R_0^1 d_1 + d_0$$

Rigid Motions: Rotation and Translation

$$p_0 = R_0^1 R_1^2 p_2 + R_0^1 d_1 + d_0$$

$$R_0^2 = R_0^1 R_1^2$$

$$d_2 = R_0^1 d_1 + d_0$$



Homogeneous Transformations

- Rewrite rigid motion as:

$$\begin{bmatrix} R_0^1 & d_0^1 \\ \varnothing & 1 \end{bmatrix} \begin{bmatrix} R_1^2 & d_1^2 \\ \varnothing & 1 \end{bmatrix} = \begin{bmatrix} R_0^1 R_1^2 & R_0^1 d_1^2 + d_0^1 \\ \varnothing & 1 \end{bmatrix}$$

  - where Ø denotes the null or zero row vector [0,0,0].

Homogeneous Transformations

- Represent the augmented transformation matrix as:

$$H = \begin{bmatrix} R & d \\ \varnothing & 1 \end{bmatrix}, R \in \mathrm{SO}(3), d \in \mathbb{R}^3$$

  - Transformation matrices of this form are called homogeneous transformations.
  - They represent both rotation and translation, $H \in SE(3)$.

Homogeneous Transformations

- What is the inverse transformation?

$$H = \begin{bmatrix} R & d \\ \varnothing & 1 \end{bmatrix}, R \in SO(3), d \in \mathbb{R}^3$$

$$\Downarrow$$

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ \varnothing & 1 \end{bmatrix}, R \in SO(3), d \in \mathbb{R}^3$$

Basic homogeneous transformations

Translations

$$Trans_{x,\Delta x} = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_{y,\Delta y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_{z,\Delta z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations

$$Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot_{y,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot_{z,\gamma} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation and rotation in a unified form

- General form of the homogeneous transformation matrix

$$H_0^1 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{d} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\mathbf{n}$ represents the direction of x_1in Frame 0 ($\mathbf{n} \rightarrow$ normal)
- $\boldsymbol{s}$ represents the direction of y_1 in Frame 0 ($\mathbf{s} \rightarrow$ sliding)
- $\boldsymbol{a}$ represents the direction of z_1 in Frame 0 ($\boldsymbol{a} \rightarrow$ approach)
- $\boldsymbol{d}$ represents the distance from the origin of Frame 0 to the origin of Frame 1

What if we need to translate and rotate?

- Combinations of homogeneous transformations:

$$H_0^2 = H_0^1 H_1^2$$

Multiplication in order!

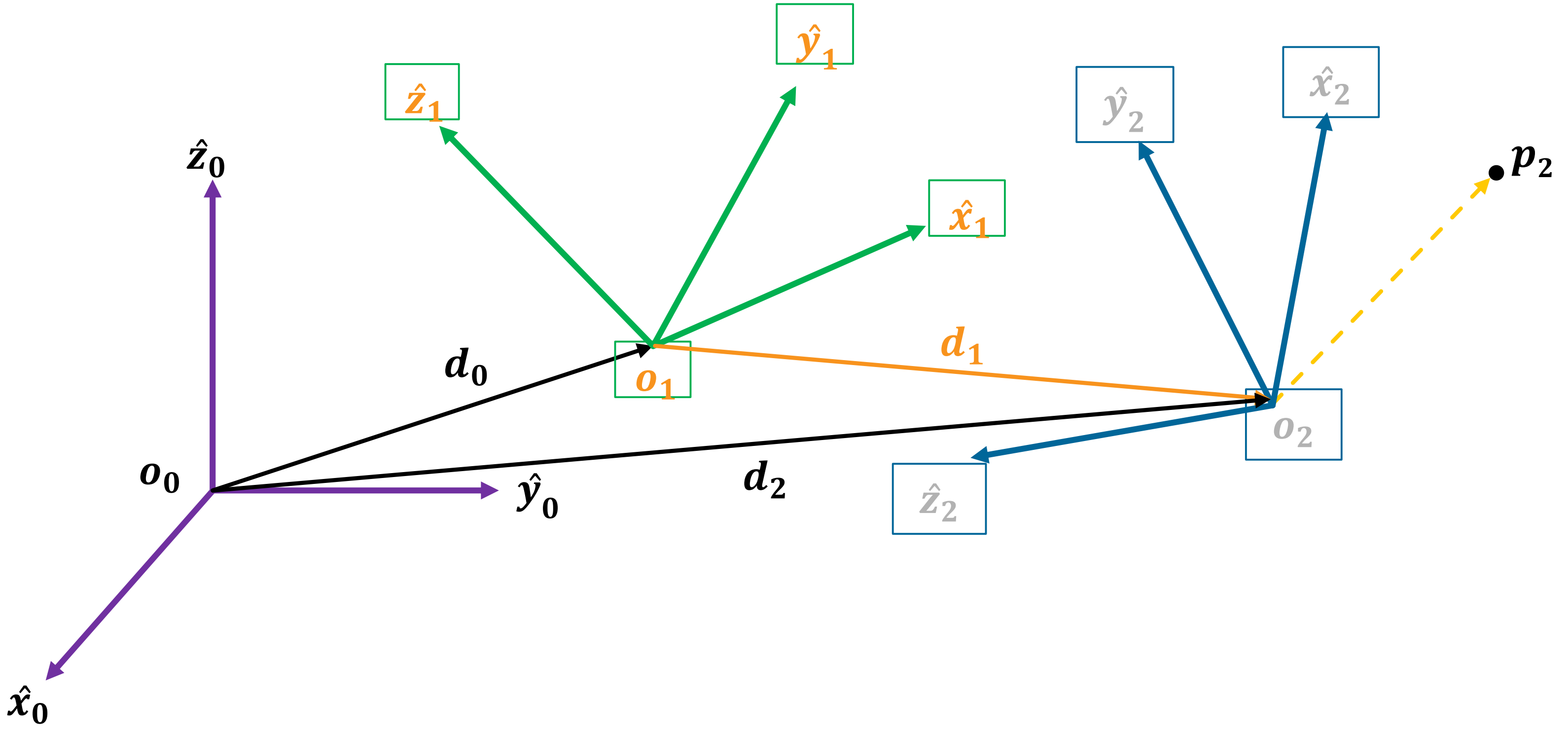
Transformation w.r.t. fixed and moving frame

What if we need to translate and rotate?

- Combinations of homogeneous transformations:

$$H_0^2 = H_0^1 H_1^2$$

Multiplication in order!



Basic homogeneous transformations

Translations

$$Trans_{x,\Delta x} = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_{y,\Delta y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_{z,\Delta z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations

$$Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

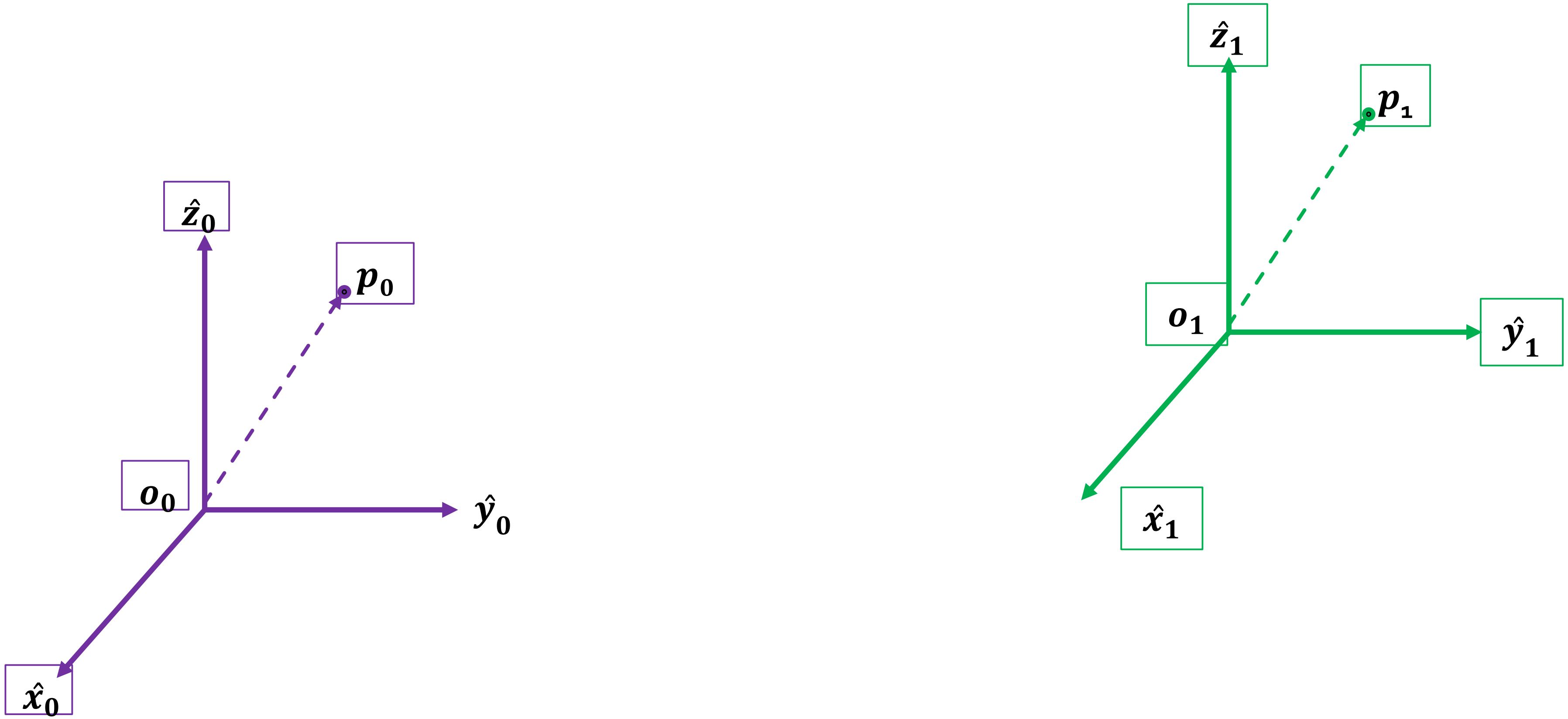
$$Rot_{y,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot_{z,\gamma} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

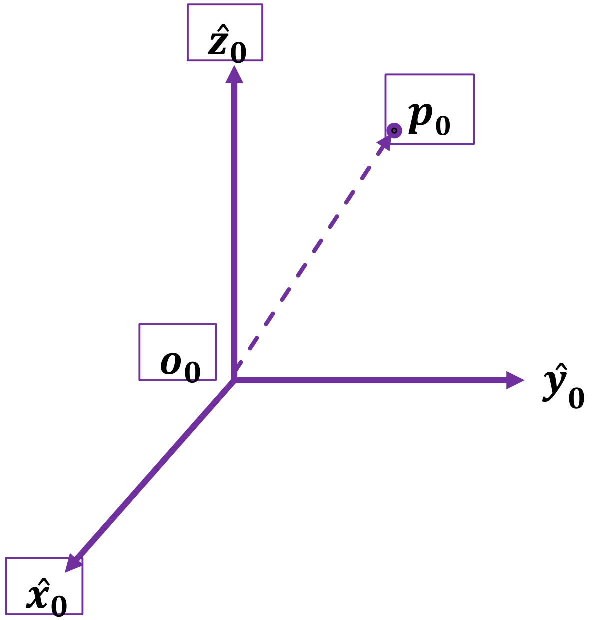
Example 1

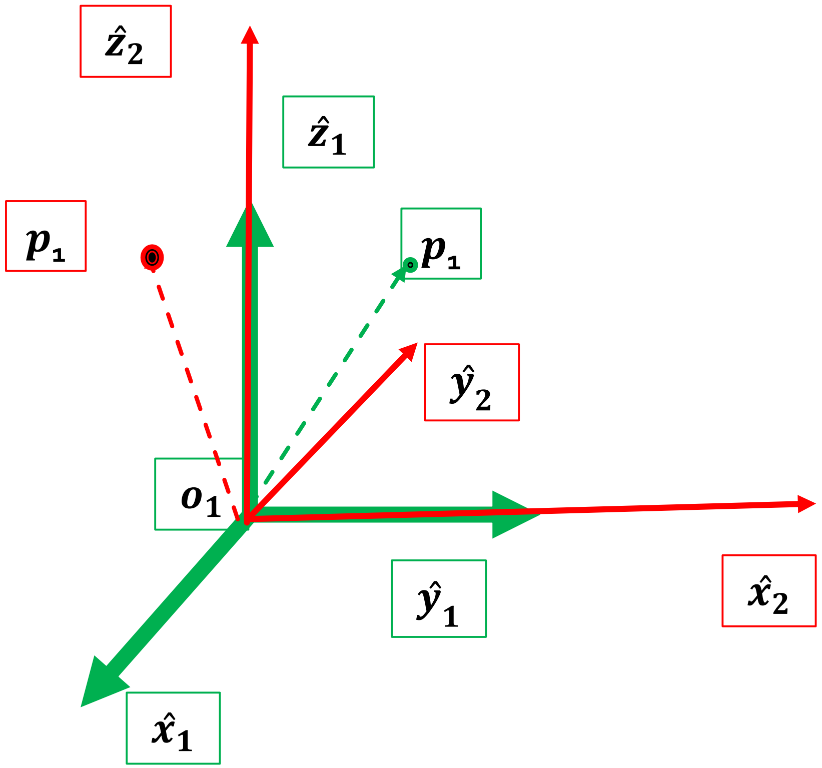
- A point P is defined as $P = [2, 3, 5]^T$ relative to Frame 0.
- Calculate the position of the point **w.r.t. the original Frame 0** after the following **transformations of Frame 0**:
  - Translate 5 units along x, 1 unit along y, and 6 units along z
  - Rotate 90 degrees about the z axis
  - Rotate 90 degrees about the y axis

Translate 5 units along x, 1 unit along y, and 6

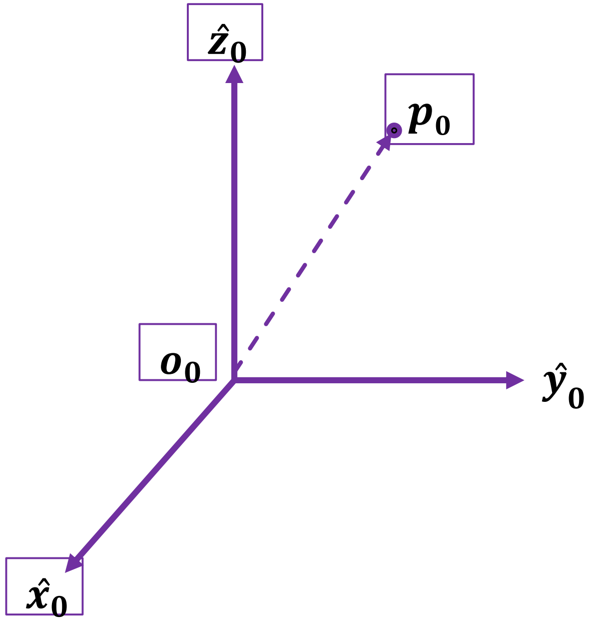


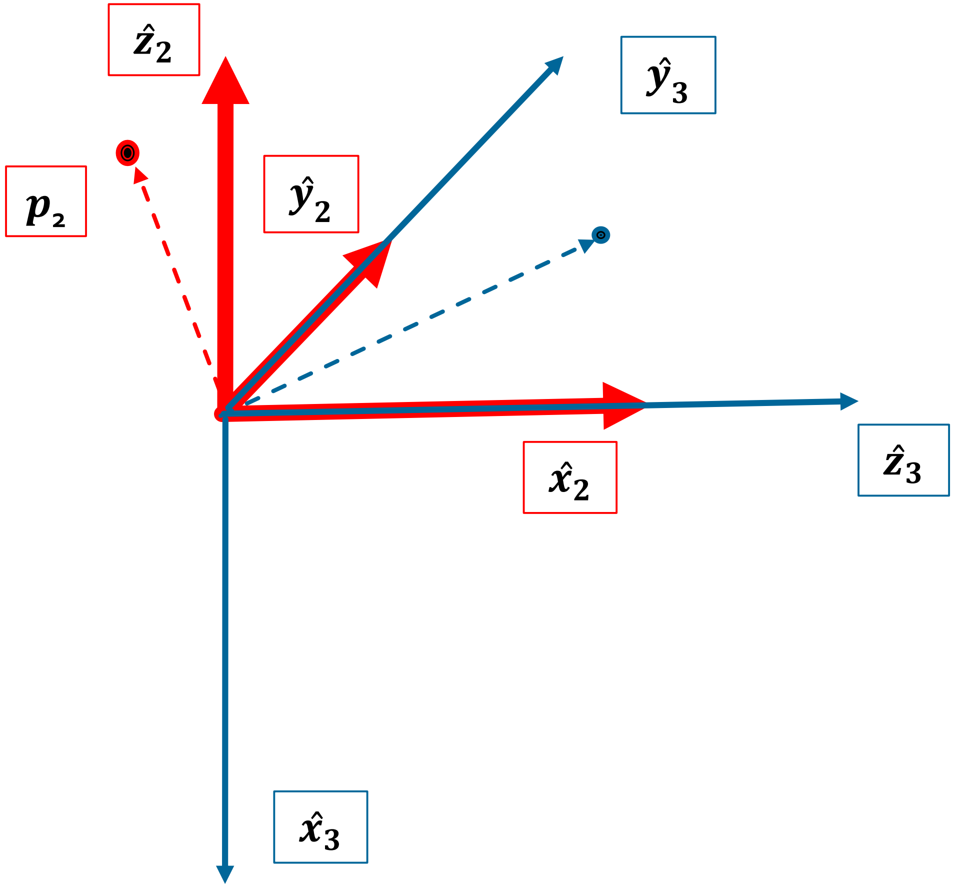
Rotate 90 degrees about the z axis





Rotate 90 degrees about the y axis





Solution

- Given $P = [2,3,5]^T$
- Translate 5 units along x, 1 unit along y, and 6 units along z

$$H_0^1 = Trans(5,1,6)$$

- Rotate 90 degrees about the z axis

$$H_1^2 = Rot_z(90^\circ)$$

- Rotate 90 degrees about the y axis

$$H_2^3 = Rot_y(90^\circ)$$

- Finally,

$$\boxed{P_{new} = H_0^3 \cdot P = H_0^1 H_1^2 H_2^3 \cdot P}$$

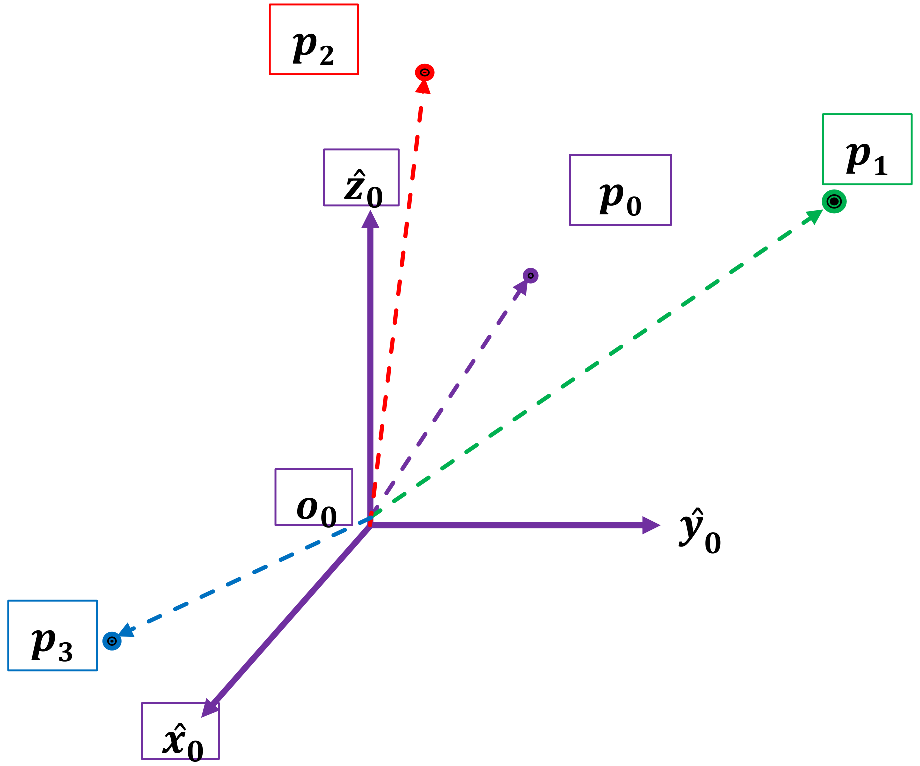
Example 2

- A point P is defined as $P = [2,3,5]^T$ relative to Frame 0.

- Calculate the position of the point after the following transformations **about the axes of original Frame 0**:
  - Translate 5 units along x, 1 unit along y, and 6 units along z
  - Rotate 90 degrees about the z axis
  - Rotate 90 degrees about the y axis

- Write your answer w.r.t the basic homogeneous transformations

Transformation w.r.t. the fixed Frame Fo



Solution

- Given $P = [2, 3, 5]^T$
- Translate 5 units along x, 1 unit along y, and 6 units along z

$$P_1 = H_0^1 \cdot P = Trans(5, 1, 6) \cdot P$$

- Rotate 90 degrees about the z axis

$$P_2 = H_1^2 \cdot P_1 = Rot_z(90°) \cdot Trans(5, 1, 6) \cdot P$$

- Rotate 90 degrees about the y axis

$$P_3 = H_2^3 \cdot P_2 = Rot_y(90°) \cdot Rot_z(90°) \cdot Trans(5, 1, 6) \cdot P$$

- Finally,

$$\boxed{P_{new} = H_2^3 H_1^2 H_0^1 \cdot P}$$

Solution

$$P_{final} = Rot(y, 90^\circ) \cdot Rot(z, 90^\circ) \cdot Trans(5,1,6) \cdot P$$

$$P_{final} = \begin{bmatrix} \cos(90) & 0 & \sin(90) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(90) & 0 & \cos(90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(90) & -\sin(90) & 0 & 0 \\ \sin(90) & \cos(90) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \\ 1 \end{bmatrix}$$

- Compute the P_{final}

Rotate a frame about X, Y, Z-axis

- Rotating about a fixed frame
- Rotating about a moving frame

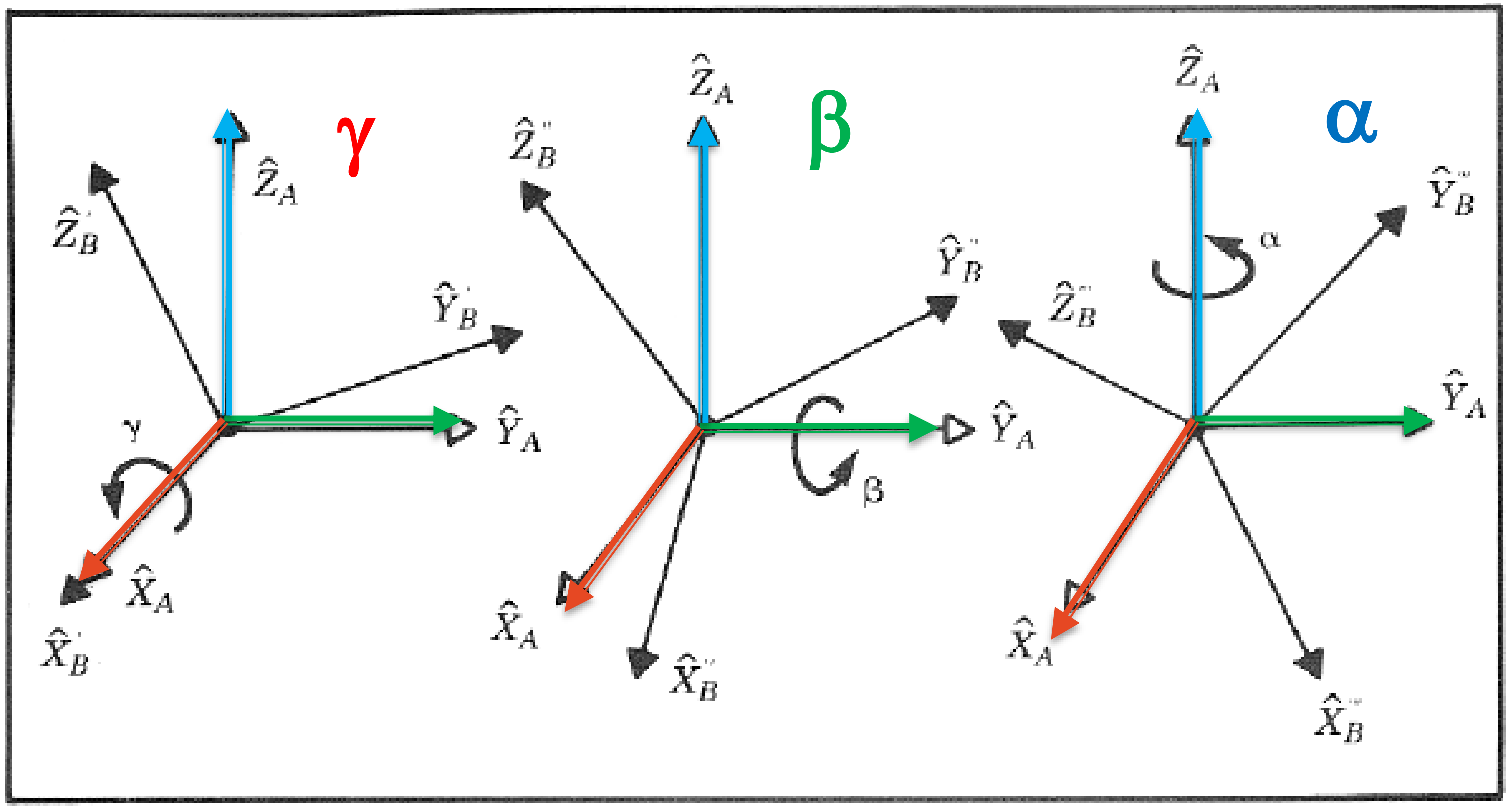
$$R_X(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$$R_Y(\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}$$

$$R_Z(\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

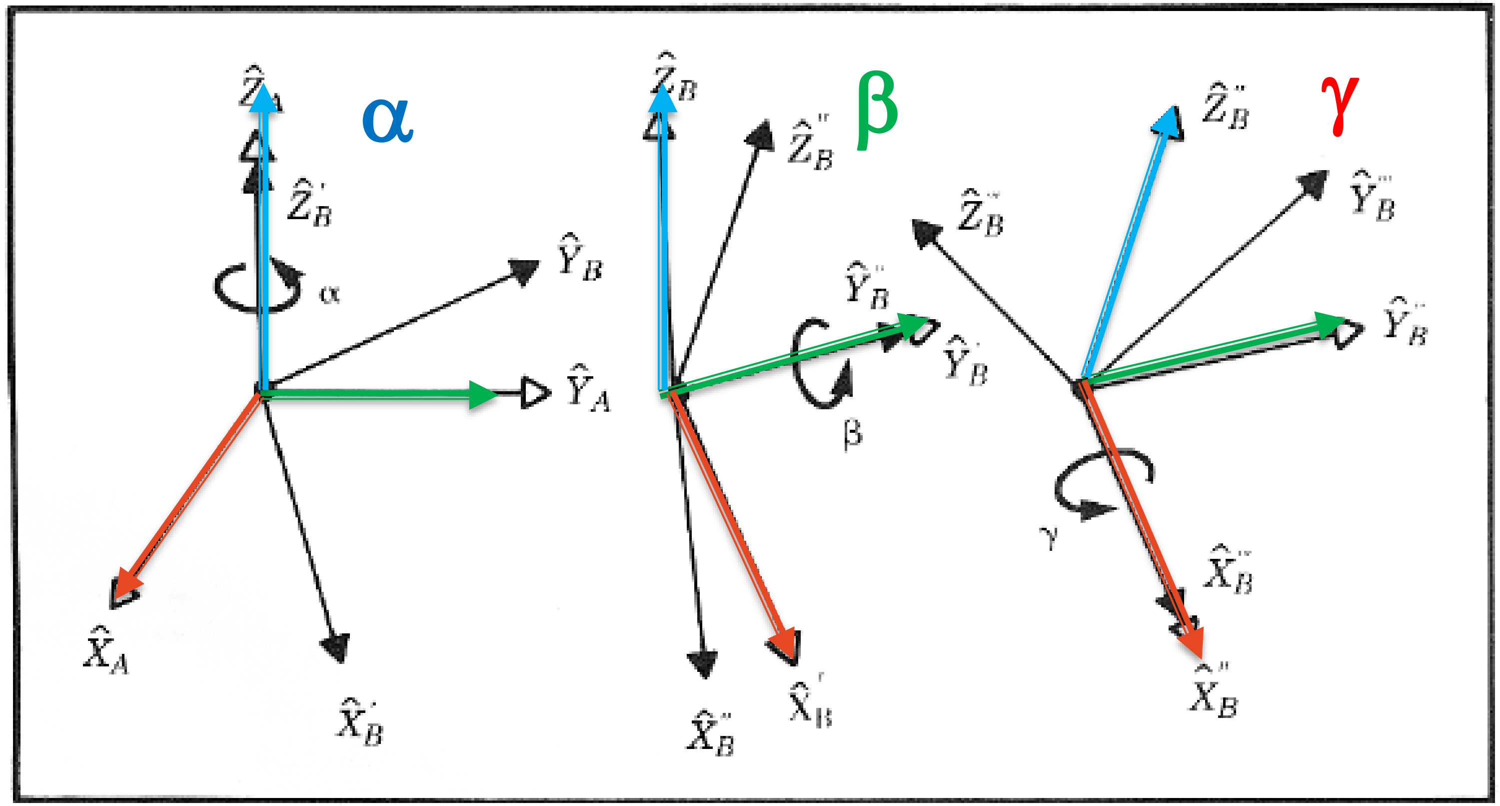
Rotating about a fixed frame

$${}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma))$$



Rotating about a moving frame

$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{Z}(\alpha)R_{Y}(\beta)R_{X}(\gamma) $$



Equivalent rotation

Rotate about a fixed frame

$${}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma))$$

$${}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$$

Rotate about a moving frame

End