Coordinate Frames

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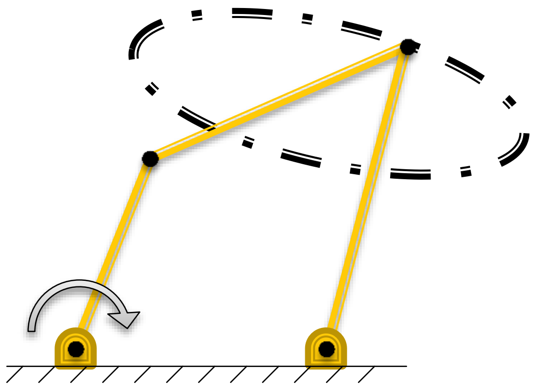
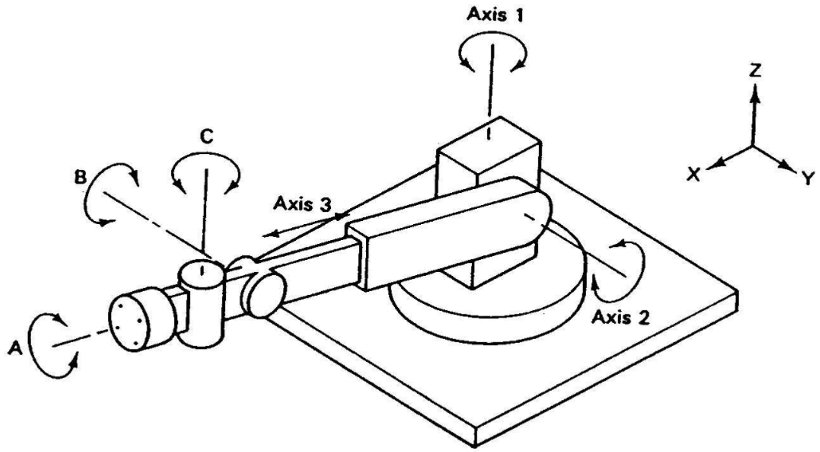
Quiz (10 pts)

- (3 pts) How many DOFs does the four-bar mechanism have?
- (3 pts) List two advantages of parallel kinematic machines?
- (4 pts) Describe the work envelop of a RRP robot

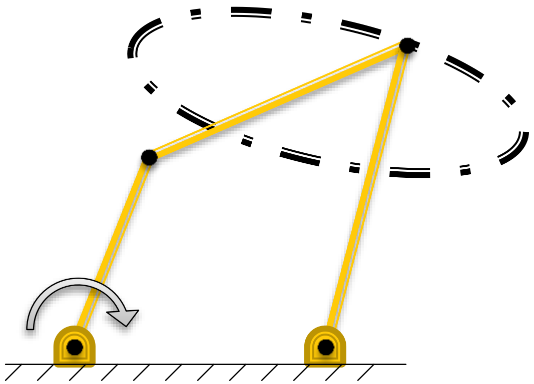




Degrees of Freedom (DOF)

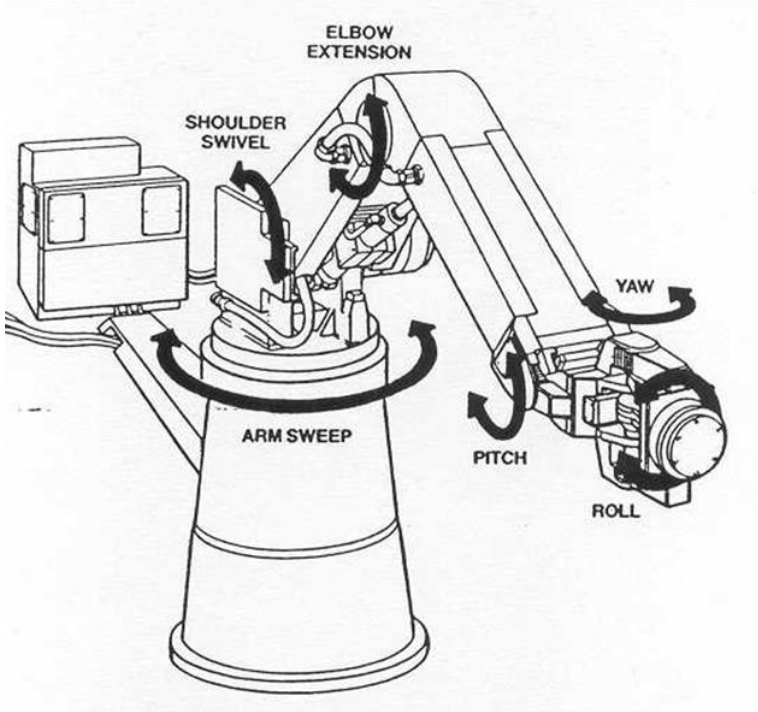
- The minimum number of required independent coordinates to completely specify robot motions
  - # of required actuators?
  - # of joints?

Degrees of Freedom (DOF)



Four Bar Linkage Mechanism

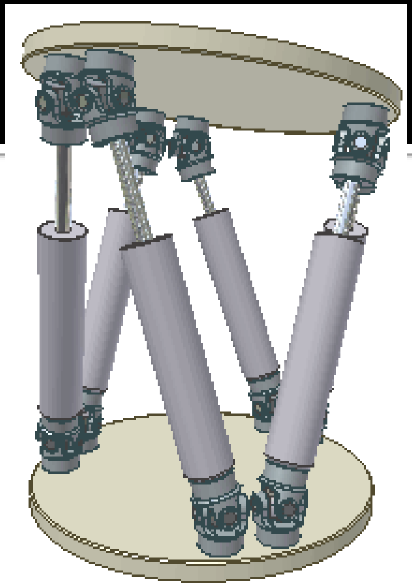
Closed-Loop Kinematic Chain



Industrial Robot

Open-Loop Kinematic Chain

Parallel Kinematics

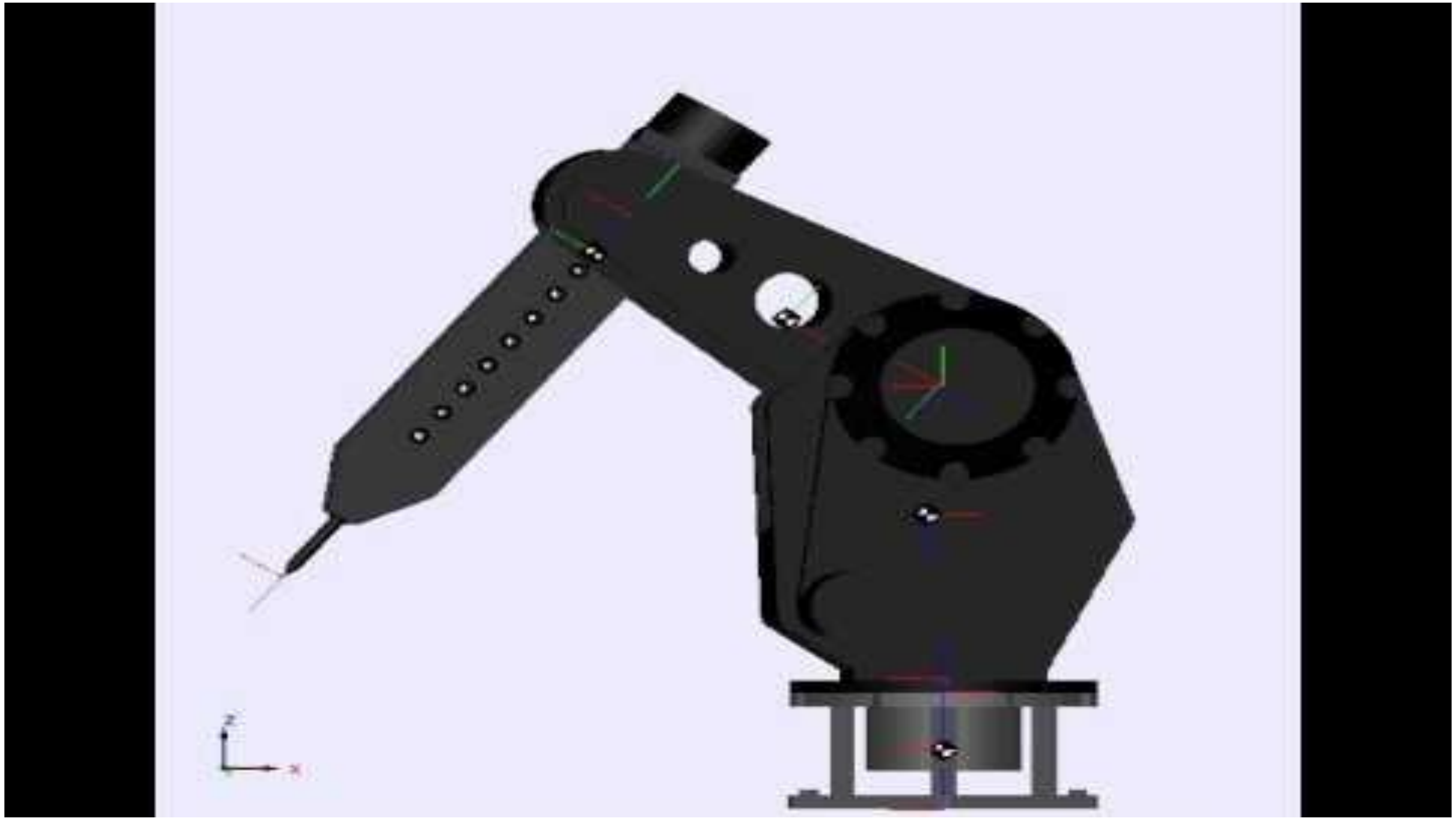


- Parallel Kinematic Machines (PKMs)
  - Closed kinematic loops
  - Stewart Platform / hexapods

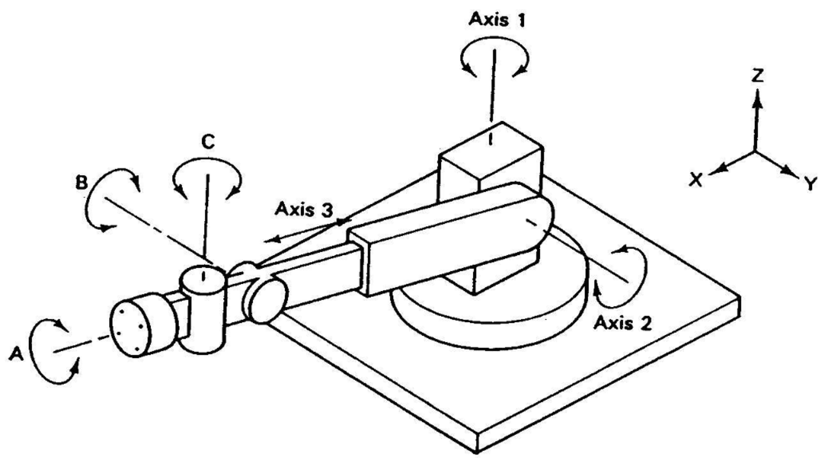
- Pros:
  - Greater rigidity – parallel links
  - Higher speed – less mass to move
  - Higher accuracy – averaged error

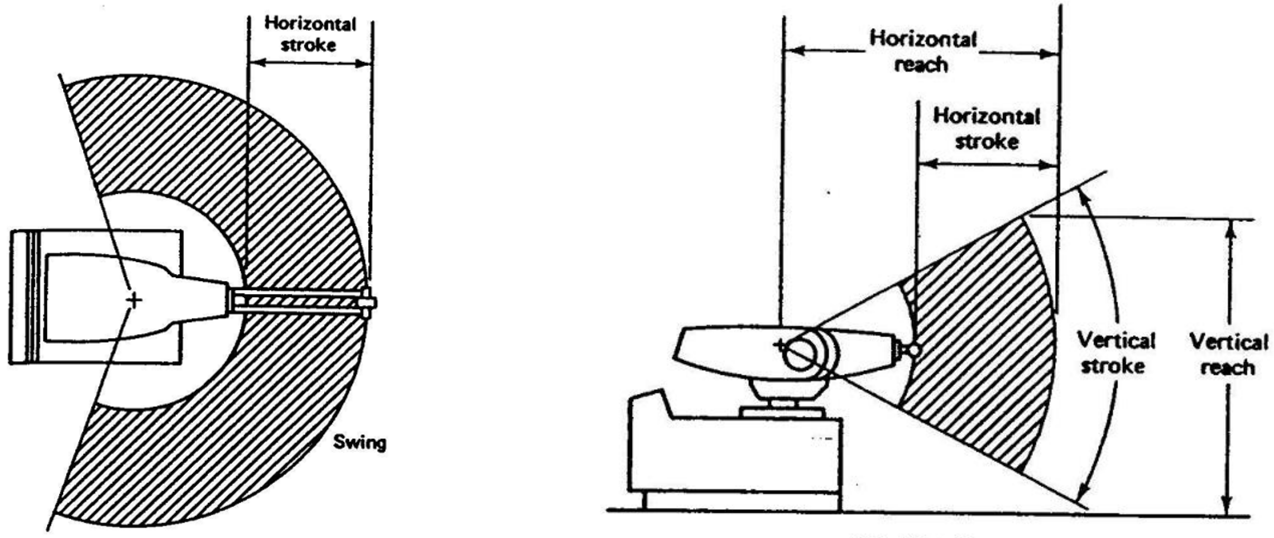
- Cons:
  - Limited work envelope
  - Requires a large space for large motion
  - Inability to avoid objects

Spherical - RRP



Work Envelope of a Spherical Robot





(a) Plan

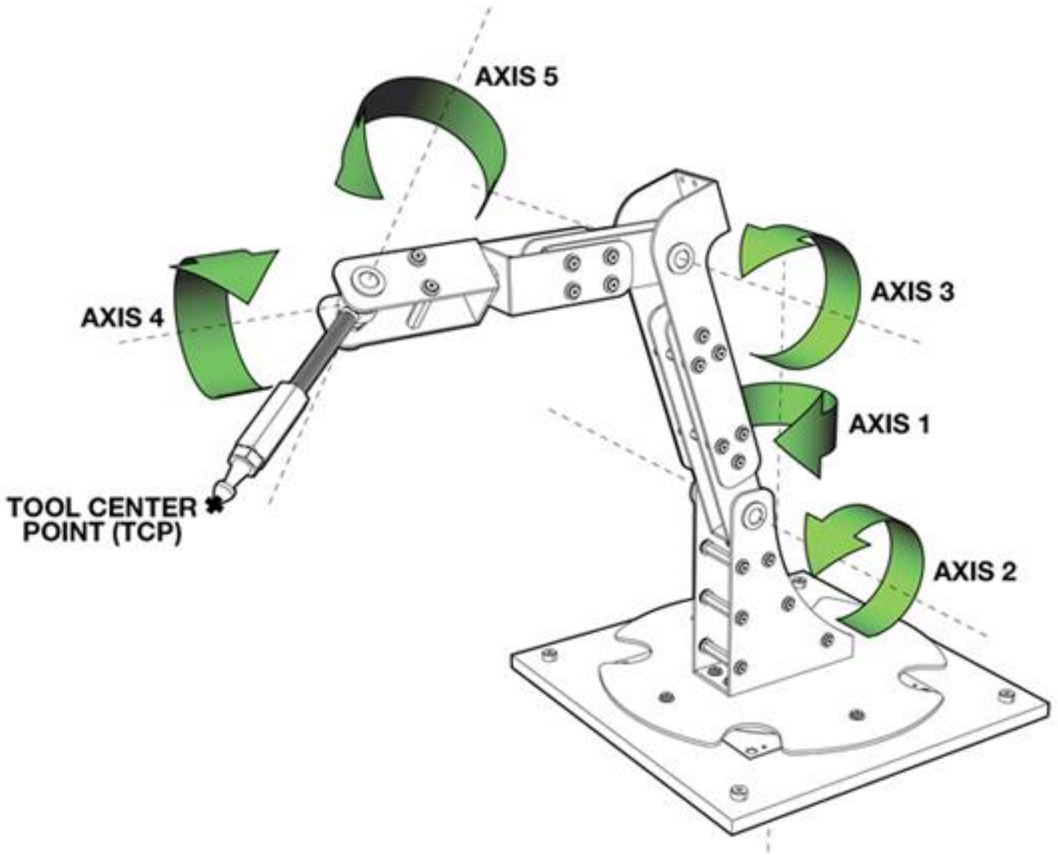
(b) Elevation

2D Transformation

Robot kinematics

- Kinematics analysis
  - Study robot motion (position, velocity, acceleration) without considering the force/torque that cause the motions

How to represent robot position in 2D/3D workspace?

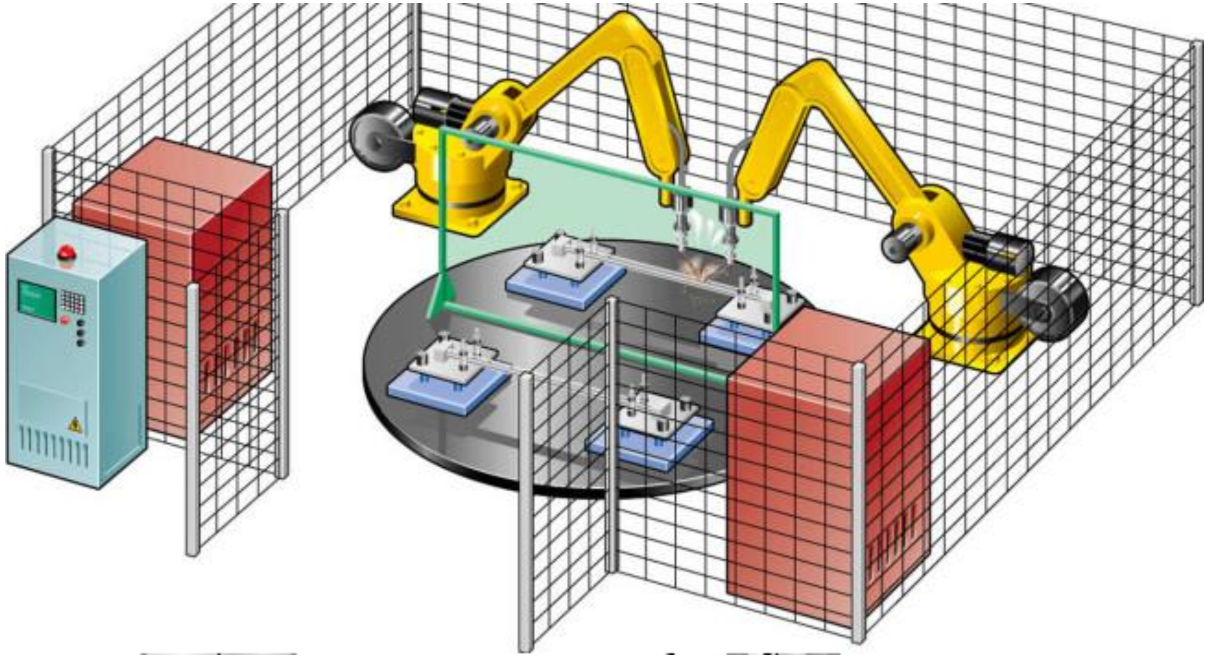


Overview

- Mathematical representations of robot position & orientation
  - Reference frame
  - Using vector to represent robot position
  - Using matrix to represent robot orientation
  - 2D homogeneous transformation

Reference frames

- Industrial robot typically operates in a “work cell”
  - World frame = the reference frame attached to the robot workspace



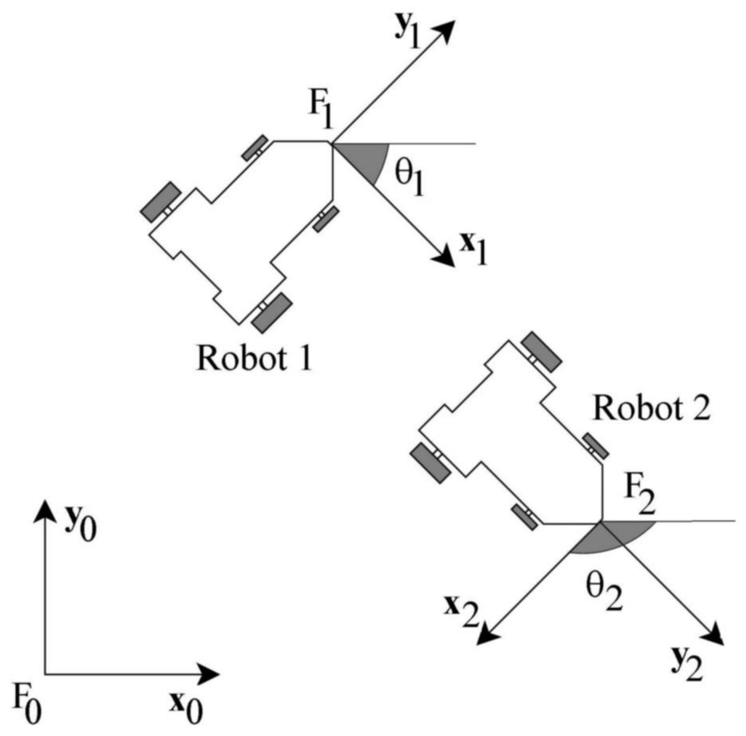
Reference frame

- Given a reference frame, you can
  - Use a vector to specify robot position
  - Use a matrix to specify robot orientation

Consider a 2D workspace

Planar Location

- World frame = Fo
- Robot frame
  - F1 attached to Robot 1
  - F2 attached to Robot 2

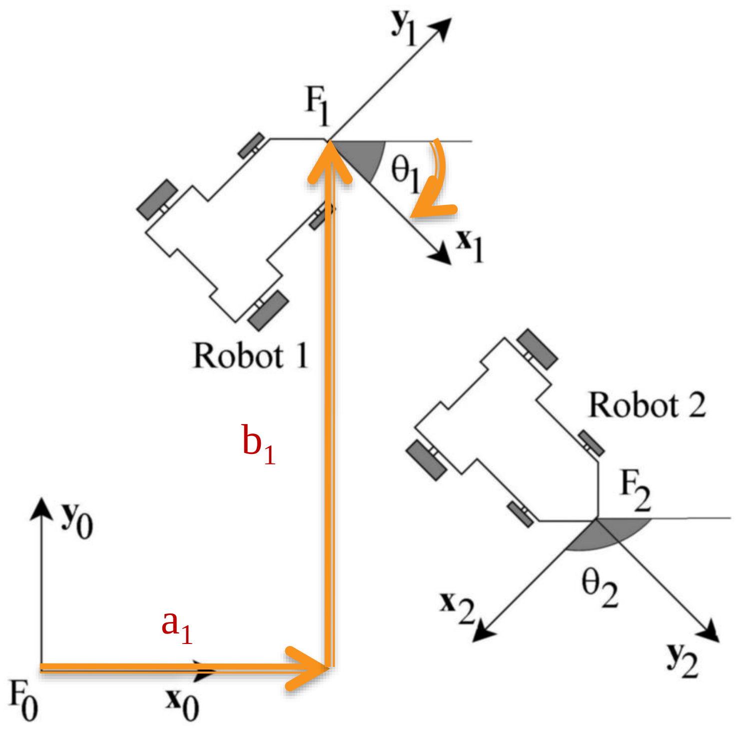


Planar Location

- Variables for describing the planar location of the robots

- Position
  - Robot 1 – (a_1, b_1) in Fo

- Orientation
  - Robot 1 – θ_1 in Fo



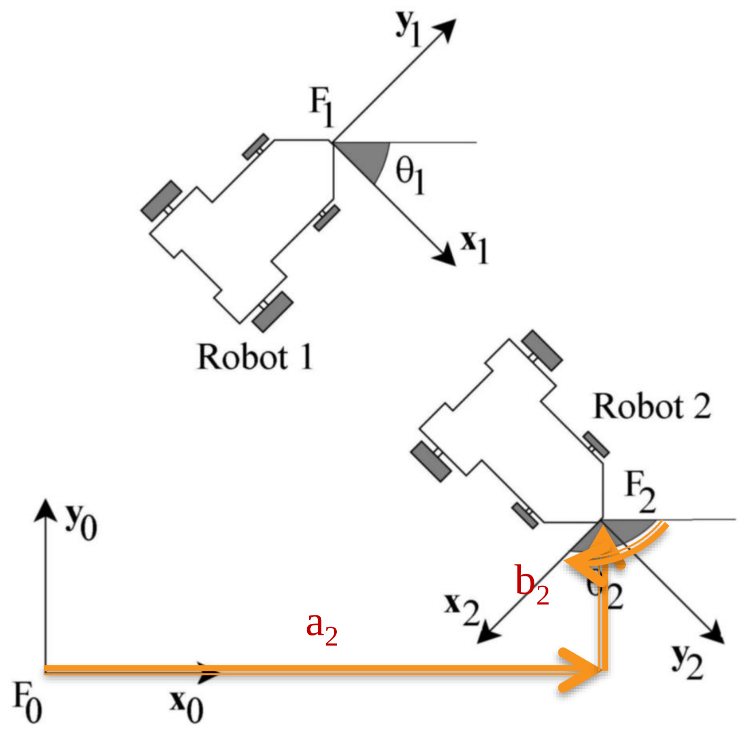
Planar Location

- Variables for describing the planar location of the robots

- Position
  - Robot 2 – (a_2, b_2) in Fo

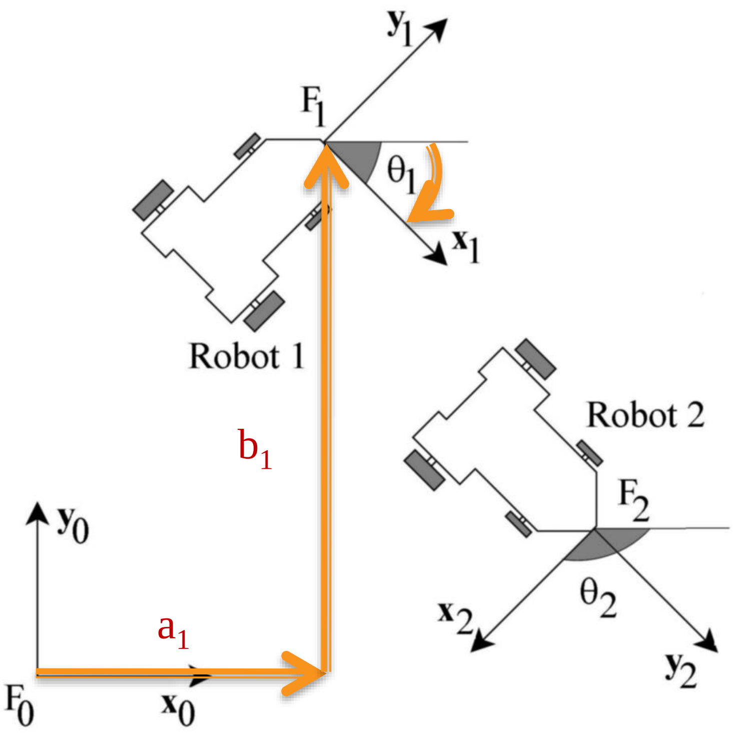
- Orientation
  - Robot 2 – θ_2 in Fo

How to use vectors and matrices?



Robot position & orientation w.r.t. world frame

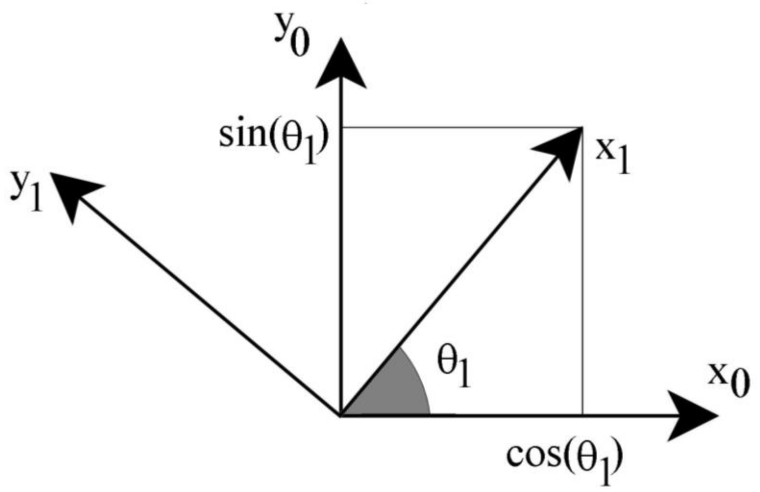
- Position of Robot 1
  - Consider the frame of robot and world frame
- Robot orientation w.r.t. Fo
  - Matrix rotation between frame orientation
  - Frame rotation
- Robot position w.r.t Fo
  - Vector difference between frame origins
  - Frame translation



Frame Rotation

- The unit vectors $\mathbf{x_1}$ and $\mathbf{y_1}$ w.r.t. F_0 are related to the angle θ_1 as follows:

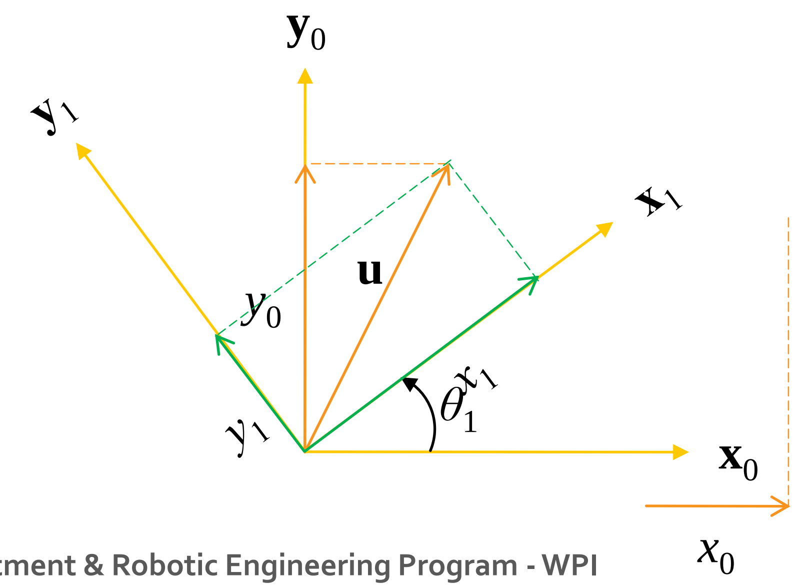
$$\mathbf{x}_1 = \cos(\theta_1)\mathbf{x}_0 + \sin(\theta_1)\mathbf{y}_0$$
$$\mathbf{y}_1 = -\sin(\theta_1)\mathbf{x}_0 + \cos(\theta_1)\mathbf{y}_0$$



Planar Orientation

- Suppose we have a vector **u** given by its coordinates w.r.t. frame F_1: $\mathbf{u} = x_1\mathbf{x}_1 + y_1\mathbf{y}_1$
  - What would the coordinates of the same point in space be relative to the F_0 coordinate system (which has the same origin as F_1 but is rotated by θ_1)?

$$\mathbf{u} = \boxed{?}\,\mathbf{x}_0 + \boxed{?}\,\mathbf{y}_0$$

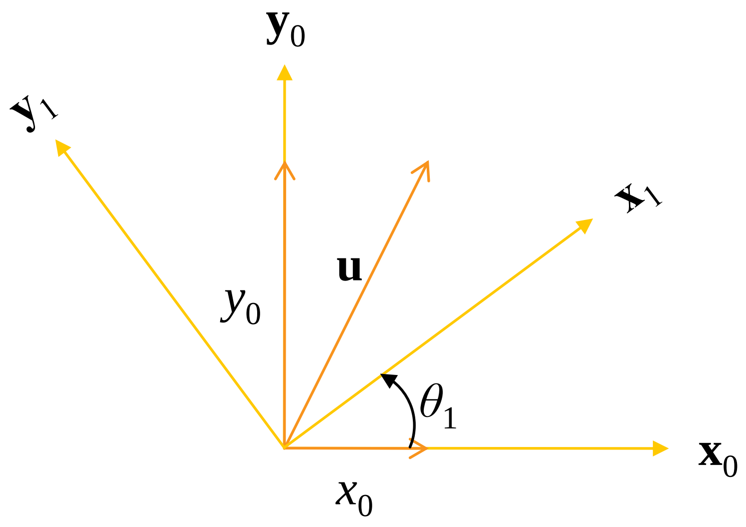


Planar Orientation

$$\mathbf{u} = x_1\mathbf{x}_1 + y_1\mathbf{y}_1$$

$$\mathbf{u} = \underbrace{x_1[\cos(\theta_1)\mathbf{x}_0 + \sin(\theta_1)\mathbf{y}_0]}_{\mathbf{x}_1} + \underbrace{y_1[-\sin(\theta_1)\mathbf{x}_0 + \cos(\theta_1)\mathbf{y}_0]}_{\mathbf{y}_1}$$

$$\mathbf{u} = [x_1\cos(\theta_1) - y_1\sin(\theta_1)]\mathbf{x}_0 + [x_1\sin(\theta_1) + y_1\cos(\theta_1)]\mathbf{y}_0$$



Planar Orientation

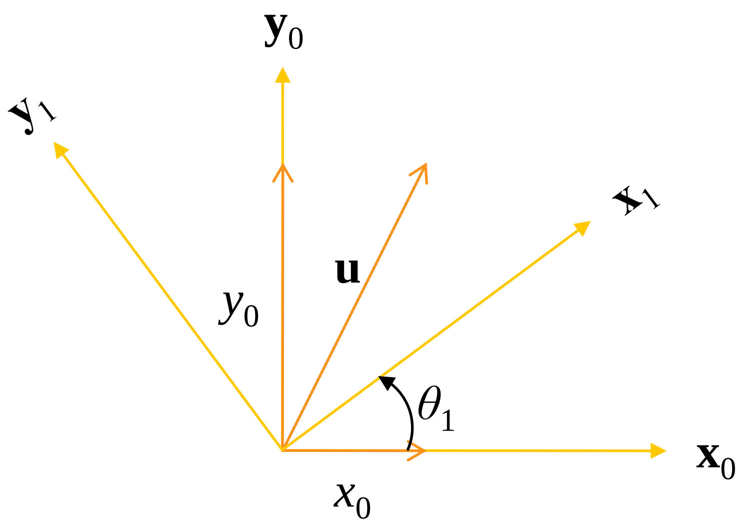
$$\mathbf{u} = [x_1\cos(\theta_1) - y_1\sin(\theta_1)]\mathbf{x}_0 + [x_1\sin(\theta_1) + y_1\cos(\theta_1)]\mathbf{y}_0$$

$$\mathbf{u} = x_0\mathbf{x}_0 + y_0\mathbf{y}_0$$

$$x_0 = x_1\cos(\theta_1) - y_1\sin(\theta_1)$$

$$y_0 = x_1\sin(\theta_1) + y_1\cos(\theta_1)$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

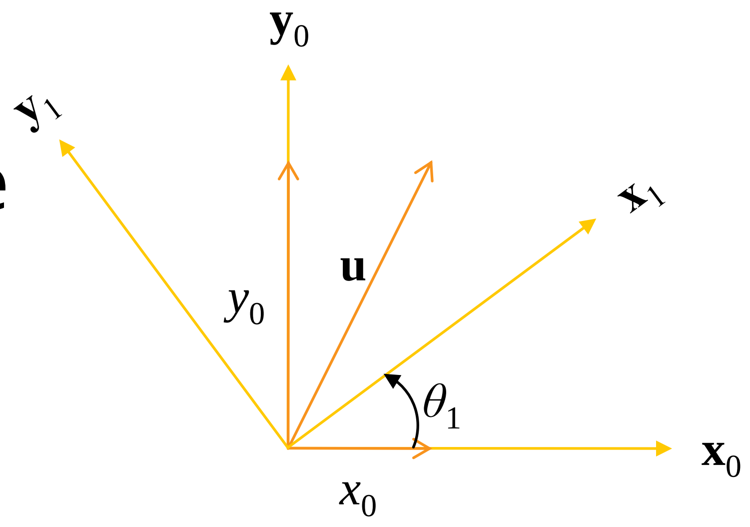


2D Rotation matrix

- The 2×2 matrix $\mathbf{R}$ is called the *rotation matrix*:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$

- The rotation matrix allows you to rotate $\mathbf{x}_0$ into $\mathbf{x}_1$ and $\mathbf{y}_0$ into $\mathbf{y}_1$

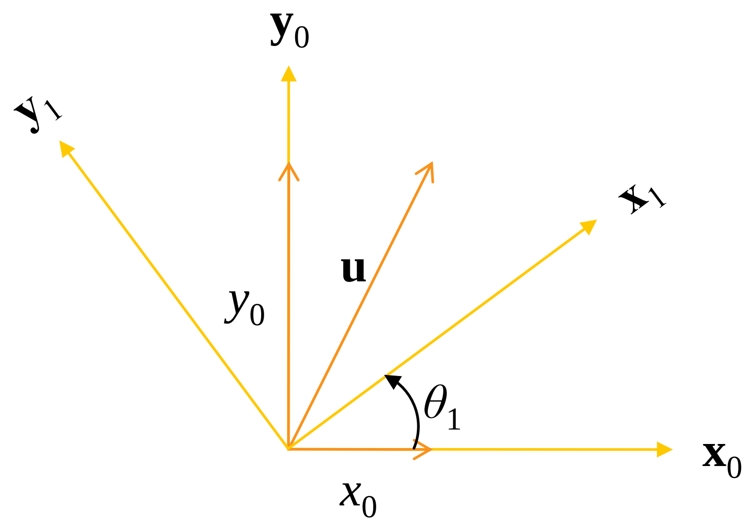


- It depends only on θ_1

2D Rotation matrix

- The rotation matrix $\mathbf{R}$ determines the orientation of frame F_1 w.r.t. F_0

- The dual nature of $\mathbf{R}$
  - Representation of a frame orientation
  - Geometric transform of rotation

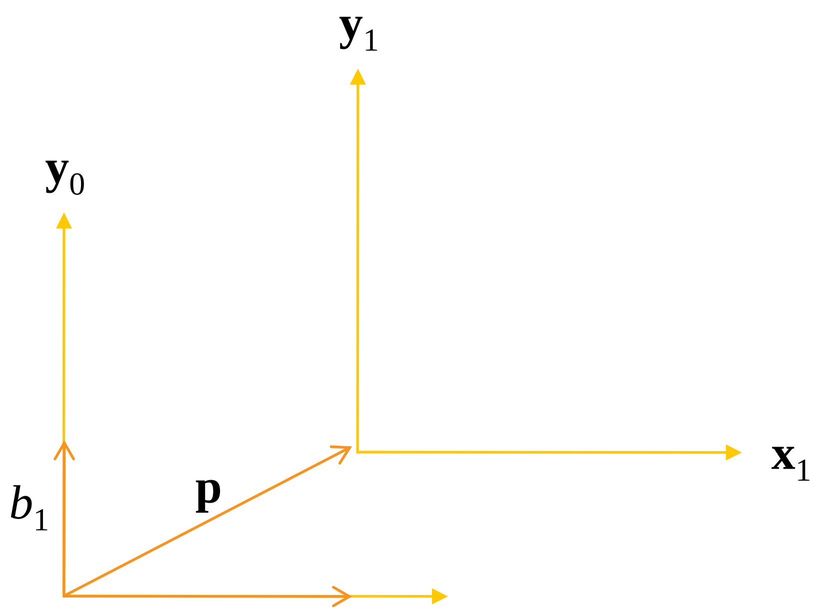


Planar position

- Vector $\mathbf{p}$ describe a translation of the F_1 coordinate system relative to the F_0 coordinate system

$$\mathbf{p} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

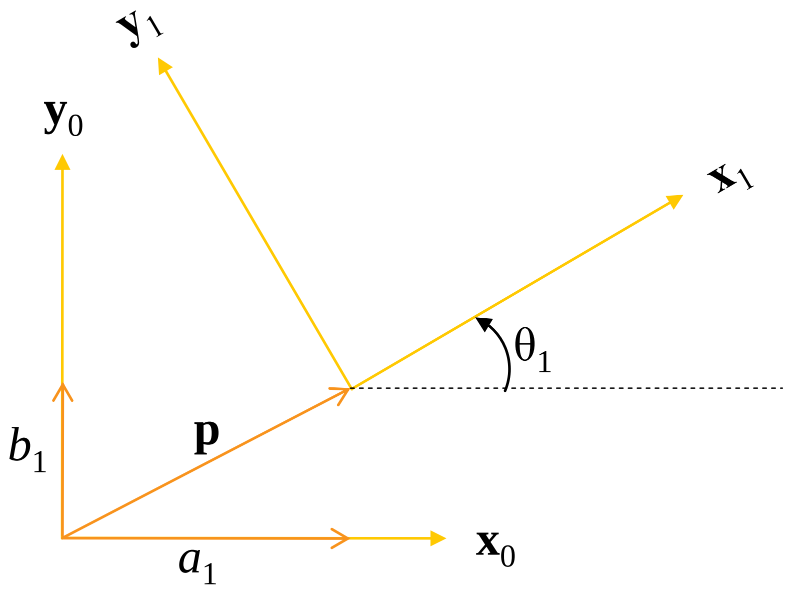
- The dual nature of $\mathbf{p}$
  - Representation of a frame position
  - Geometric transform of translation



Homogeneous transformation

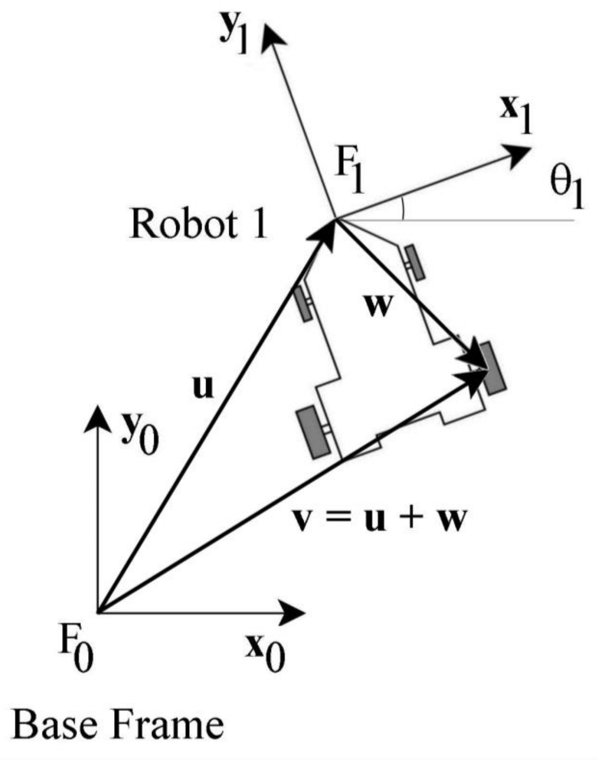
- Combine both the rotation matrix **R** with the position vector **p**
- Describe the planar pose (position + orientation) of F_1 (relative to F_0) in a single 3×3 matrix **P**

$$\mathbf{P} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & a_1 \\ \sin(\theta_1) & \cos(\theta_1) & b_1 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

- Given that
  - The center of the right rear wheel is at planar coordinates (0.4, -0.73) w.r.t. frame F_1
  - a_1 = 5, b_1 = 2, and θ_1 = 45°
- What are the coordinates of the wheel w.r.t. frame F_0?

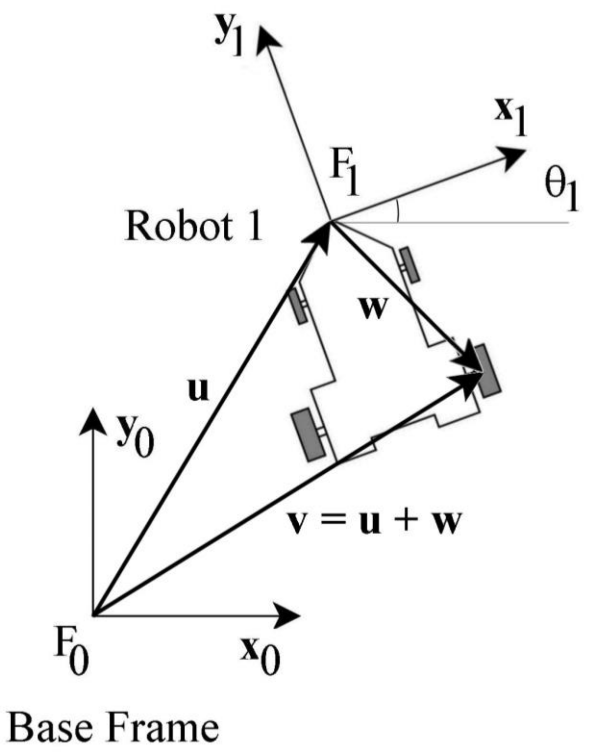


Example

- Let x_w = 0.4 and y_w = -0.73 be the coordinates of the wheel in frame F_1

- The vector **w** where the wheel is located (in frame F_1) is given by Eq. 1:

$$\mathbf{w} = \begin{bmatrix} x_w \\ y_w \end{bmatrix} = x_w \mathbf{x}_1 + y_w \mathbf{y}_1 \qquad (1)$$



Example

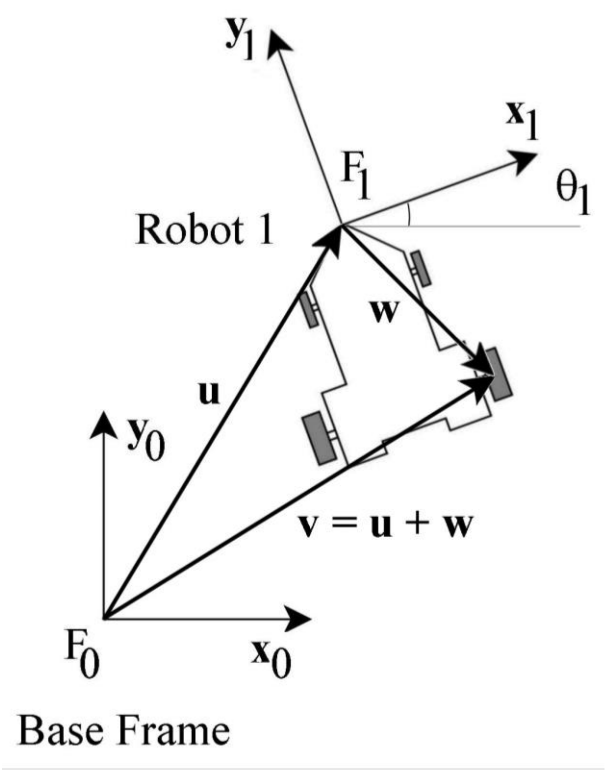
- Remember

$$\mathbf{u} = x_1\mathbf{x}_1 + y_1\mathbf{y}_1$$

$$\mathbf{u} = x_1\underbrace{[\cos(\theta_1)\mathbf{x}_0 + \sin(\theta_1)\mathbf{y}_0]}_{\mathbf{x}_1} + y_1\underbrace{[-\sin(\theta_1)\mathbf{x}_0 + \cos(\theta_1)\mathbf{y}_0]}_{\mathbf{y}_1}$$

- Similarly,

$$\mathbf{w} = \left[x_w\cos(\theta_1) - y_w\sin(\theta_1)\right]\mathbf{x}_0 + \left[x_w\sin(\theta_1) + y_w\cos(\theta_1)\right]\mathbf{y}_0 \qquad (2)$$



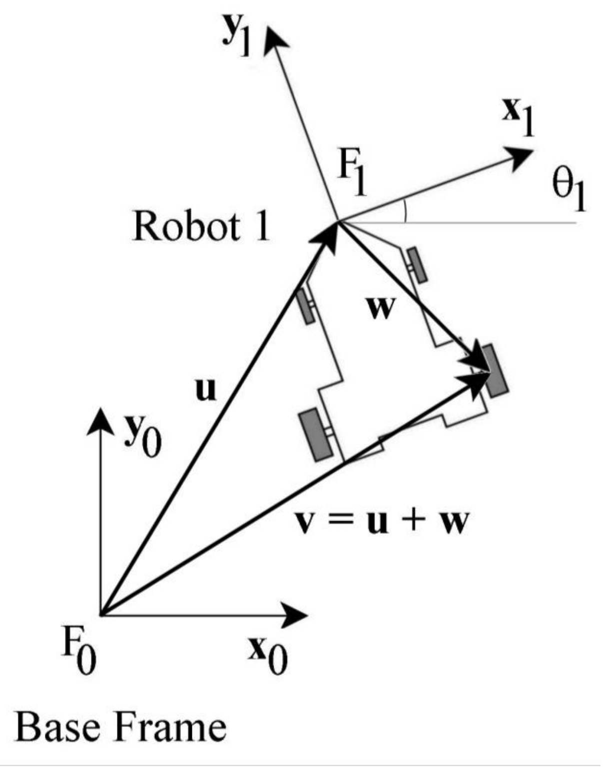
Example

- Now we have

$$ {}^{1}\mathbf{w} = \begin{bmatrix} x_w \\ y_w \end{bmatrix} \qquad {}^{0}\mathbf{w} = \begin{bmatrix} x_w \cos(\theta_1) - y_w \sin(\theta_1) \\ x_w \sin(\theta_1) + y_w \cos(\theta_1) \end{bmatrix} $$

- Sum of vectors

$$ \mathbf{v} = \mathbf{u} + \mathbf{w} = \begin{bmatrix} a_1 + x_w \cos(\theta_1) - y_w \sin(\theta_1) \\ b_1 + x_w \sin(\theta_1) + y_w \cos(\theta_1) \end{bmatrix} $$

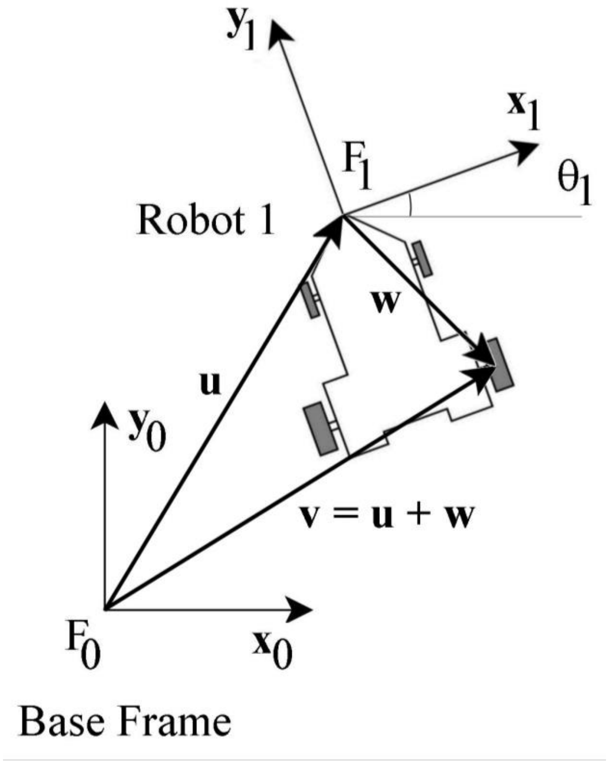


Alternative solution?

- A more efficient way to solve the problem is to use the combined matrix **P**:

$$\mathbf{v} = \mathbf{P}\mathbf{w} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & a_1 \\ \sin(\theta_1) & \cos(\theta_1) & b_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + x_w \cos(\theta_1) - y_w \sin(\theta_1) \\ b_1 + x_w \sin(\theta_1) + y_w \cos(\theta_1) \\ 1 \end{bmatrix}$$



Homogeneous free vectors

- These are free vectors in that they don't represent a point in space but rather a direction (they are unit vectors)

- Homogeneous *free* vectors are formed by the addition of a zero-element

- For example, the unit vector $\mathbf{x}_1$ has the homogeneous

value of $^{1}\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in frame F_1, and $^{0}\mathbf{x}_1 = \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{bmatrix}$ in frame F_0

-

Homogeneous free vectors

- The matrix $\mathbf{P}$ (also called *pose matrix*)as a generalized representation of the location of frame F_1 (w.r.t. F_0)

- Each column vectors of matrix $\mathbf{P}$ consists of a homogeneous vector

Free vectors

Point vector

$$\mathbf{P} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & a_1 \\ \sin(\theta_1) & \cos(\theta_1) & b_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{p}_1 \end{bmatrix}$$

Homogeneous free vectors

- The third row of **P** as well as of homogeneous vectors (0 for free, and 1 for point) ensures that the distinction between free and point vectors is maintained through the transformation

$$\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & a_1 \\ \sin(\theta_1) & \cos(\theta_1) & b_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{bmatrix}$$

$$\mathbf{P} \quad {}^{1}\mathbf{x}_1 = {}^{0}\mathbf{x}_1$$

Apply transformation to a homogeneous free vector

Homogeneous free vectors

- The third row of $\mathbf{P}$ as well as of homogeneous vectors (0 for free, and 1 for point) ensures that the distinction between free and point vectors is maintained through the transformation

$$\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & a_1 \\ \sin(\theta_1) & \cos(\theta_1) & b_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 + x_w \cos(\theta_1) - y_w \sin(\theta_1) \\ b_1 + x_w \sin(\theta_1) + y_w \cos(\theta_1) \\ 1 \end{bmatrix}$$

$$\mathbf{P} \quad {}^1w = {}^0w$$

Apply transformation to a homogeneous point vector

End