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The Null Spaces of Elliptic Partial Differential Operators in $\mathbb{R}^n$

LOUIS NIRENBERG*

Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012

AND

HOMER F. WALKER

Department of Mathematics, Texas Tech University, P.O. Box 4319, Lubbock, Texas 79409

Submitted by P. D. Lax

1. INTRODUCTION

For each real number p, $1 \leqslant p < \infty$, let $L_p(\mathbb{R}^n; \mathbb{C}^k)$ denote the usual Banach space of equivalence classes of $\mathbb{C}^k$-valued functions on $\mathbb{R}^n$ whose absolute values raised to the power p are Lebesgue integrable over $\mathbb{R}^n$. For each positive integer m, let $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ denote the Banach space consisting of those elements of $L_p(\mathbb{R}^n; \mathbb{C}^k)$ which have (strong) partial derivatives or order m in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. Denote the usual norm on $L_p(\mathbb{R}^n; \mathbb{C}^k)$ by $\| \ \|_p$, and take

$$\| u \|_{m,p} = \left\{ \sum_{|\alpha| \leqslant m} \left\| \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p^p \right\}^{1/p}$$

to be the norm on $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$, the notation being standard multiindex notation. In the following, each linear partial differential operator,

$$Au(x) = \sum_{|\alpha| \leqslant m} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x),$$

of order m is assumed to have domain $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ in $L_p(\mathbb{R}^n; \mathbb{C}^k)$ and to have $k \times k$ coefficient matrices continuous in x on $\mathbb{R}^n$. The partial differential operators of particular interest are those which are *elliptic* in the sense that

$$\det \left| \sum_{|\alpha| = m} a_\alpha(x)\, \xi^\alpha \right| \neq 0$$

for all x in $\mathbb{R}^n$ and all nonzero ξ in $\mathbb{R}^n$.

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For a given integer m, consider a linear elliptic partial differential operator

$$A_\infty u(x) = \sum_{|\alpha|=m} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

of order m which has constant coefficients and no terms of order less than m. It is shown in [11], a paper generalizing results in [7] and [10], that if there is given a second elliptic operator A of order m, whose coefficients converge at infinity uniformly to those of A_∞ with a certain swiftness, then the dimension of the null space $N_2(A)$ of A in $H_{m,2}(\mathbb{R}^n; \mathbb{C}^k)$ is finite. Furthermore, it is shown that the dimension of $N_2(A)$ depends upper-semicontinuously on such an operator A in the sense that the dimension of the null-space of the operator does not increase if the coefficients of the operator are perturbed slightly inside a ball of finite radius about the origin in $\mathbb{R}^n$. In this paper, the Fourier-transform technique of [11] is abandoned in favor of an approach based on fundamental solutions. The principal results of [11] mentioned here are extended in the following not only to include a broader class of elliptic operators which are allowed to be perturbed in a less restricted manner but also to allow the domains of these operators to be any of the spaces $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$, $1 < p < \infty$. In order to demonstrate that the theorems obtained are very nearly the best possible, an example is presented of an elliptic operator with an infinite-dimensional null space. This example is sharper than the one given in [11].

The following conventions are used throughout the sequel "large" and "small" constants whose only important property is their size are denoted generically by C and ϵ, respectively. Constants which are otherwise distinguished are subscripted. The numbers p, p' always satisfy

$$1 < p < \infty; \qquad (1/p) + (1/p') = 1.$$

$L_p(\mathbb{R}^n; \mathbb{C}^1)$ and $H_{m,p}(\mathbb{R}^n; \mathbb{C}^1)$ are denoted by $L_p(\mathbb{R}^n)$ and $H_{m,p}(\mathbb{R}^n)$, respectively.

2. Preparatory Results for Operators with Constant Coefficients

In this and the next section we establish two theorems which are used in the sequel to study the null spaces of the elliptic operators of interest. The L_p-estimates described in these theorems have well-known analogs in similar investigations concerning elliptic operators on a compact set or manifold. The first theorem relies on the two lemmas below, which are of interest in themselves.

LEMMA 2.1. *For real numbers a and b whose sum is positive, consider the kernel*

$$K(x, y) = \frac{1}{|x|^a |x - y|^{n-a-b} |y|^b}, \qquad \text{for } x \neq y \text{ in } \mathbb{R}^n.$$

The integral operator

$$Ku(x) = \int_{\mathbb{R}^n} K(x, y)\, u(y)\, dy$$

is a bounded operator on $L_p(\mathbb{R}^n)$ if and only if $a < n/p$ and $b < n/p'$.

Proof. To show that the conditions $a < n/p$ and $b < n/p'$ are necessary, suppose that K is a bounded operator on $L_p(\mathbb{R}^n)$. Then the functions

$$v(x) = \int_{|y| \leqslant 1} K(x, y)\, dy \qquad \text{and} \qquad w(y) = \int_{|x| \leqslant 1} K(x, y)\, dx$$

are in $L_p(\mathbb{R}^n)$ and $L_{p'}(\mathbb{R}^n)$, respectively. For large $|x|$ and $|y|$, $v(x)$ behaves like a constant multiple of $|x|^{-n+b}$ and $w(y)$ behaves like a constant multiple of $|y|^{-n+a}$. Then $v(x)$ belongs to $L_p(\mathbb{R}^n)$ and $w(y)$ belongs to $L_{p'}(\mathbb{R}^n)$ only if $p(n - b) > n$ and $p'(n - a) > n$, i.e., only if $b < n/p'$ and $a < n/p$.

To show that the conditions $a < n/p$ and $b < n/p'$ are sufficient, note first that it may be assumed without loss of generality that both a and b are nonnegative. (If, say, a is negative, then b is positive, and from the inequality

$$\frac{|x|}{|x - y|} \leqslant 1 + \frac{|y|}{|x - y|}$$

it is seen that

$$K(x, y) \leqslant \frac{C}{|x - y|^{n-b} |y|^b} + \frac{C}{|x - y|^{n-a-b} |y|^{a+b}}.$$

The boundedness of K in this case then follows immediately from the result in the case in which both a and b are nonnegative.) Now for nonnegative a and b satisfying $a < n/p$ and $b < n/p'$, a comparison of arithmetic and geometric means yields the inequality

$$|x|^n \geqslant \prod_{i=1}^{n} |x_i|,$$

and it follows that

$$K(x, y) \leqslant \prod_{i=1}^{n} \frac{1}{|x_i|^{a/n} |x_i - y_i|^{1-(a+b)/n} |y_i|^{b/n}}.$$

In the light of this last inequality, it is apparent that the desired result follows from the corresponding one-dimensional statement: If a and b are nonnegative numbers with positive sum satisfying $a < 1/p$ and $b < 1/p'$, then the integral operator

$$Ku(x) = \int_{\mathbb{R}^1} K(x, y)\, u(y)\, dy$$

with kernel

$$K(x, y) = \frac{1}{|x|^a\, |x - y|^{1-a-b}\, |y|^b}$$

defined on $\mathbb{R}^1 \times \mathbb{R}^1$, is a bounded operator on $L_p(\mathbb{R}^1)$. This one-dimensional result is, in turn, an easy consequence of the following well-known lemma (see [4], Theorem 3.9).

LEMMA. *Suppose that $K(x, y)$ is nonnegative and homogeneous of degree (-1) for $x \geqslant 0$ and $y \geqslant 0$, and that the (necessarily identical) quantities*

$$\int_0^\infty K(x, 1)\, x^{-1/p'}\, dx \qquad \textit{and} \qquad \int_0^\infty K(1, y)\, y^{-1/p}\, dy$$

are equal to some number $C < \infty$. Then the integral operator

$$Ku(x) = \int_0^\infty K(x, y)\, u(y)\, dy$$

is bounded on $L_p((0, \infty))$ with norm no greater than C.

This completes the proof of Lemma 2.1.

LEMMA 2.2. *Let $\Omega(x)$ be an infinitely differentiable function defined on $\mathbb{R}^n - \{0\}$ which does not vanish identically, which is homogeneous of degree zero, and which satisfies*

$$\int_{|x|=1} \Omega(x)\, dw = 0,$$

where dw is the element of volume on the unit sphere in $\mathbb{R}^n$. For a real number a, consider the function

$$K(x, y) = \frac{\Omega(x - y)}{|x|^a\, |x - y|^n\, |y|^{-a}}\,.$$

The integral operator

$$Ku(x) = \int_{\mathbb{R}^n} K(x, y)\, u(y)\, dy$$

is a bounded operator on $L_p(\mathbb{R}^n)$ *if and only if* $-n/p' < a < n/p$.

The necessity of the condition $-n/p' < a < n/p$ is easily proved by an argument similar to that in Lemma 2.1; the assertion that this condition is sufficient is a special case of the theorem in [9].

As in the introduction, let A_∞ denote a given linear elliptic partial differential operator with constant coefficients which is homogeneous of order m. The theorem below describes an estimate involving this operator which plays a fundamental role in the investigations that follow.

THEOREM 2.1. *Let* r *be the smallest nonnegative integer greater than* $(m - n/p')$, *and let* ρ *be a number satisfying the following conditions*

$$\begin{gathered} -\frac{n}{p} < \rho < r - m + \frac{n}{p'}; \\ \rho + m - \frac{n}{p'} \text{ is not a nonnegative integer.} \end{gathered} \tag{2.1}$$

Then there exists a constant C_0 *for which the estimate*

$$\sum_{|\alpha| \leqslant m} \left\| \, | x |^{|\alpha| + \rho} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \leqslant C_0 \| \, | x |^{m+\rho} A_\infty u \, \|_p \tag{2.2}$$

holds for all u *in* $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ *such that* $| x |^{m+\rho} A_\infty u$ *is in* $L_p(\mathbb{R}^N; \mathbb{C}^k)$. *Furthermore, if* ρ *does not satisfy the condition* (2.1), *then no inequality having the form of* (2.2) *can hold.*

Note that, in particular, inequality (2.2) holds with $\rho = 0$ if and only if $(m - n/p')$ is not a nonnegative integer.

Proof. (a) It will first be shown that if ρ does not satisfy the conditions (2.1), then no inequality having the form of (2.2) can hold. The case in which $\rho \leqslant -n/p$ can be dispensed with immediately: If $\rho \leqslant -n/p$ and if u is any function in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ which has compact support and which is equal to some nonzero constant in a neighborhood of the origin, then, for this function u, the quantity on the right side of (2.2) is finite while the quantity on the left side is infinite. In treating the remaining cases, use will be made of the fundamental solution of A_∞ of the form

$$\Gamma(x) = \Gamma_0(x) + \log | x | \, \Gamma_1(x), \tag{2.3}$$

where Γ_0 and Γ_1 are homogeneous of degree $(m - n)$ (see [5], [6], and others for information concerning the existence and properties of fundamental solutions of elliptic operators).

To treat the case in which $\rho \geqslant r - m + n/p'$, let σ be an infinitely differentiable real-valued function on $\mathbb{R}^n$ satisfying $0 \leqslant \sigma(x) \leqslant 1$ for all x in $\mathbb{R}^n$, $\sigma(x) = 0$ for $|x| \leqslant 1$, and $\sigma(x) = 1$ for $|x| \geqslant 2$. If one defines

$$u(x) = \sigma(x) \frac{\partial^r}{\partial x_1{}^r} \Gamma(x)$$

for x in $\mathbb{R}^n$, then there exists a constant C such that

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} u(x) \right| \leqslant C \, | x |^{m-n-r-|\alpha|} \log | x | \qquad \text{for } | x | \geqslant 2, \ | \alpha | \leqslant m.$$

Since $m - n - r < -n/p$, it is evident that u is in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$. Furthermore, $A_\infty u(x) = 0$ for $|x| \geqslant 2$, and so the quantity on the right side of (2.2) is finite for this function u. But for an appropriate constant C, the inequality

$$| x |^\rho \, | u(x) | \geqslant C \, | x |^{m-n-r+\rho}$$

holds for $|x| \geqslant 2$ and x in some open cone with vertex at the origin. Consequently, if $\rho \geqslant r - m + n/p'$, the quantity on the left side of (2.2) is infinite for this function u.

To treat the final case in which $-n/p < \rho < r - m + n/p'$ and $\rho + m - n/p' = s$, a nonnegative integer, let $\{\sigma_R\}_{3 \leqslant R < \infty}$ be a collection of infinitely differentiable real-valued functions satisfying the following conditions:

(i) For each $R \geqslant 3$, $0 \leqslant \sigma_R(x) \leqslant 1$ for all x in $\mathbb{R}^n$, $\sigma_R(x) = 0$ for $|x| \leqslant 1$, $\sigma_R(x) = 1$ for $2 \leqslant |x| \leqslant R$, and $\sigma_R(x) = 0$ for $|x| \geqslant 2R$;

(ii) There exists a constant C, independent of R, such that the inequalities

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \sigma_R(x) \right| \leqslant C R^{-|\alpha|}, \qquad R \leqslant | x | \leqslant 2R,$$

and

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \sigma_R(x) \right| \leqslant C, \qquad 1 \leqslant | x | \leqslant 2,$$

hold for $|\alpha| \leqslant m$ and $R \geqslant 3$.

(Such a collection of functions is easily constructed.) For small positive ϵ, define

$$u_R(x) = \sigma_R(x) \mid x \mid^{-\epsilon} \frac{\partial^s}{\partial x_1^{\,s}} \Gamma(x)$$

for each $R \geqslant 3$. Suppose first that $(\partial^s/\partial x_1^{\,s})\, \Gamma_1(x)$ does not vanish identically. Then it is easily seen that there exists a constant C, independent of R and ϵ, for which the following inequalities hold

$$\mid \mid x \mid^{m+\rho} A_\infty u_R(x) \mid \leqslant C \qquad \text{for } 1 \leqslant \mid x \mid \leqslant 2;$$

$$\mid \mid x \mid^{m+\rho} A_\infty u_R(x) \mid \leqslant C\epsilon \mid x \mid^{m+\rho-n-\epsilon-s} \log \mid x \mid = C\epsilon \mid x \mid^{-\epsilon-n/p} \log \mid x \mid$$
$$\text{for } 2 \leqslant \mid x \mid \leqslant R;$$

$$\begin{aligned} \mid \mid x \mid^{m+\rho} A_\infty u_R(x) \mid &\leqslant C\epsilon \mid x \mid^{m+\rho-n-\epsilon-s} \log \mid x \mid + CR^{m+\rho-n-\epsilon-s} \log R \\ &= C\epsilon \mid x \mid^{-\epsilon-n/p} \log \mid x \mid + CR^{-\epsilon-n/p} \log R \\ &\qquad \text{for } R \leqslant \mid x \mid \leqslant 2R. \end{aligned}$$

Hence, for a different constant C, independent of R and ϵ, one has the estimate

$$\int_{\mathbb{R}^n} \mid \mid x \mid^{m+\rho} A_\infty u_R(x) \mid^p dx \leqslant C + C\epsilon^p \int_2^{2R} t^{-\epsilon p-1} (\log t)^p \, dt + CR^{-\epsilon p} (\log R)^p.$$

On the other hand, the inequality

$$\begin{aligned} \int_{\mathbb{R}^n} \mid \mid x \mid^{\rho} u_R(x) \mid^p dx &\geqslant C \int_2^R t^{(\rho+m-n-\epsilon-s)p+n-1} (\log t)^p \, dt \\ &= C \int_2^R t^{-\epsilon p-1} (\log t)^p \, dt \end{aligned}$$

holds for an appropriate constant C independent of R and ϵ. Suppose that an estimate having the form of (2.2) exists. By applying this estimate and the inequality just derived to the functions $u_R(x)$ and taking limits as R approaches infinity, it follows that there exists a constant C, independent of ϵ, for which the inequality

$$\int_2^\infty t^{-\epsilon p-1} (\log t)^p \, dt \leqslant C + C\epsilon^p \int_2^\infty t^{-\epsilon p-1} (\log t)^p \, dt$$

holds for every small positive ϵ. But no such inequality can hold for every positive ϵ, since the integral that appears grows without bound as ϵ approaches zero. Hence, no estimate having the form (2.2) can exist if $(\partial^s/\partial x_1^{\,s})\, \Gamma_1$ is not identically zero. Now if $(\partial^s/\partial x_1^{\,s})\, \Gamma_1$ is identically zero and if it is assumed that an estimate having the form of (2.2) exists, then one obtains by the same argument a constant C, independent of ϵ, for which the inequality

$$\int_2^\infty t^{-\epsilon p-1}\,dt \leqslant C + C\epsilon^p \int_2^\infty t^{-\epsilon p-1}\,dt$$

holds for every small positive ϵ. As before, no such inequality can hold for arbitrarily small positive ϵ, and it must be the case that no estimate having the form of (2.2) can exist if $-n/p < \rho < r - m + n/p'$ and $\rho + m - n/p' = s$, a nonnegative integer.

(b) To prove (2.2) for ρ satisfying conditions (2.1) we shall first establish an apparently weaker estimate: There exists a constant C for which the estimate

$$\|\,|x|^\rho u\,\|_p \leqslant C\,\|\,|x|^{m+\rho} A_\infty u\,\|_p \tag{2.4}$$

holds for all u in $H_{m,p}(\mathbb{R}^n;\mathbb{C}^k)$ such that $|x|^{m+\rho} A_\infty u$ is in $L_p(\mathbb{R}^n;\mathbb{C}^k)$. Inequality (2.2) will be derived from (2.4) in part (c).

To prove (2.4), let u be an element of $H_{m,p}(\mathbb{R}^n;\mathbb{C}^k)$ and denote $A_\infty u$ by f. Assume that u is such that $|x|^{m+\rho} f$ is in $L_p(\mathbb{R}^n;\mathbb{C}^k)$. Let $\{\xi_R\}_{0<R<\infty}$ be a collection of infinitely differentiable real-valued functions satisfying the following conditions:

(i) For each positive R, $0 \leqslant \xi_R(x) \leqslant 1$ for all x in $\mathbb{R}^n$, $\xi_R(x) = 1$ for $|x| \leqslant R$, and $\xi_R(x) = 0$ for $|x| \geqslant 2R$.

(ii) There exists a constant C, independent of R, such that the inequality $|(\partial^\alpha/\partial x^\alpha)\,\xi_R(x)| \leqslant CR^{-|\alpha|}$ holds for all x in R^n and all α with $|\alpha| \leqslant m$.

If Γ is the fundamental solution of A_∞, one has for $|x| \leqslant R$:

$$\begin{aligned}
u(x) &= \int_{\mathbb{R}^n} \Gamma(x-y)\,[A_\infty(\xi_R u)]\,(y)\,dy \\
&= \int_{\mathbb{R}^n} \Gamma(x-y)\,\xi_R(y) f(y)\,dy \\
&\quad + \int_{\mathbb{R}^n} \Gamma(x-y) \sum_{\substack{|\alpha|+|\beta|=m\\ |\alpha|>0}} C_{\alpha\beta} \frac{\partial^\alpha}{\partial y^\alpha}\,\xi_R(y)\,\frac{\partial^\beta}{\partial y^\beta}\,u(y)\,dy \\
&= \int_{\mathbb{R}^n} \Gamma(x-y)\,\xi_R(y) f(y)\,dy \\
&\quad + \sum_{\substack{|\alpha|+|\beta|=m\\ |\alpha|>0}} (-1)^{|\beta|}\, C_{\alpha\beta} \int_{R\leqslant|y|\leqslant 2R} \frac{\partial^\beta}{\partial y^\beta}\left[\Gamma(x-y)\,\frac{\partial^\alpha}{\partial y^\alpha}\,\xi_R(y)\right] u(y)\,dy
\end{aligned}$$

for suitable constants $C_{\alpha\beta}$. Since u is in $L_p(\mathbb{R}^n; \mathbb{C}^n)$, one verifies easily that the last term on the right tends to zero as R approaches infinity, and hence,

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy. \tag{2.5}$$

The estimate (2.4) is easily established if $(\rho + m - n/p')$ is negative. In this case it follows from (2.1) that m is less than n; consequently the term Γ_1 in (2.3) vanishes (see [6, pp. 65–72]). It follows from (2.5) that

$$\begin{aligned} |u(x)| &\leqslant C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-m}} |f(y)|\, dy \\ &\leqslant C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-m} |y|^{m+\rho}} ||y|^{m+\rho} f(y)|\, dy \end{aligned}$$

or

$$||x|^\rho u(x)| \leqslant C \int_{\mathbb{R}^n} \frac{1}{|x|^{-\rho} |x-y|^{n-m} |y|^{m+\rho}} ||y|^{m+\rho} f(y)|\, dy$$

for some constant C independent of u. The estimate (2.4) is an immediate consequence of this inequality and Lemma 2.1, taking $a = -\rho$ and $b = m + \rho$.

To establish the estimate (2.4) for the case in which $(\rho + m - n/p')$ is positive, note that, in this case, it follows from the conditions (2.1) that r is positive; consequently $r = m - n + [n/p] + 1$, where $[n/p]$ denotes the largest integer no greater than n/p. Then, since p is greater than 1, it is seen that $(n - 1 - [n/p]) = (m - r)$ is nonnegative. Now the conditions (2.1) imply that there is a positive integer s such that

$$s - 1 < \rho + m - n/p' < s. \tag{2.6}$$

Therefore, it must be the case that s is less than or equal to r. (Otherwise, $(s-1)$ would be greater than or equal to r, in which case (2.6) would imply that r is less than $(\rho + m - n/p')$, contradicting the conditions (2.1).) Again appealing to the conditions (2,1), one sees that

$$m - n < \rho + m - n/p' < s \leqslant r \leqslant m. \tag{2.6$'$}$$

Now expansion of $\Gamma(x-y)$ in a finite Taylor series and substitution in (2.5) yields

$$u(x) = \sum_{|\alpha| \leqslant s-1} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial x^\alpha} \Gamma(x) \int_{\mathbb{R}^n} (-y)^\alpha f(y)\, dy + \int_{\mathbb{R}^n} R(x,y) f(y)\, dy,$$

where

$$R(x, y) = \Gamma(x - y) - \sum_{|\alpha| \leqslant s-1} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial x^\alpha} \Gamma(x) (-y)^\alpha$$

$$= \sum_{|\alpha| = s} \frac{s}{\alpha!} (-y)^\alpha \int_0^1 \frac{\partial^\alpha}{\partial x^\alpha} \Gamma(x - ty) (1 - t)^{s-1} \, dt.$$

Observe that the integrals $\int_{\mathbb{R}^n} (-y)^\alpha f(y) \, dy$, for $|\alpha| \leqslant s - 1$, that appear in this expansion are convergent since $|x|^{m+\rho} f(x)$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. Set

$$\tilde{u}(x) = \int_{\mathbb{R}^n} R(x, y) f(y) \, dy,$$

and suppose for the moment that $|x|^\rho \tilde{u}$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$ and that there exists a constant C, independent of u, for which the estimate

$$\| |x|^\rho \tilde{u} \|_p \leqslant C \| |x|^{m+\rho} f \|_p \tag{2.4$'$}$$

holds. Now if ρ is nonnegative and $|x|^\rho \tilde{u}$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$, then $\tilde{u}$ is an L_p function near infinity. Hence

$$u(x) - \tilde{u}(x) = \sum_{|\alpha| \leqslant s-1} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial x^\alpha} \Gamma(x) \int_{\mathbb{R}^n} (-y)^\alpha f(y) \, dy$$

is an L_p function near infinity. But one verifies using (2.3) and (2.6′) that this can happen only if

$$u(x) - \tilde{u}(x) = \sum_{|\alpha| \leqslant s-1} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial x^\alpha} \Gamma(x) \int_{\mathbb{R}^n} (-y)^\alpha f(y) \, dy = 0.$$

Similarly, if ρ is negative, it follows that

$$|x|^\rho u(x) - |x|^\rho \tilde{u}(x) = \sum_{|\alpha| \leqslant s-1} \frac{1}{\alpha!} |x|^\rho \frac{\partial^\alpha}{\partial x^\alpha} \Gamma(x) \int_{\mathbb{R}^n} (-y)^\alpha f(y) \, dy$$

is an L_p function near infinity. Again, it is seen from (2.3) and (2.6) that this is possible only if

$$u(x) - \tilde{u}(x) = \sum_{|\alpha| \leqslant s-1} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial x^\alpha} \Gamma(x) \int_{\mathbb{R}^n} (-y)^\alpha f(y) \, dy = 0.$$

The outcome of this is that $|x|^\rho \tilde{u}$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$ only if $u = \tilde{u}$, in which case the estimate (2.4)′ becomes the desired estimate (2.4). Consequently, to establish the estimate (2.4), it is sufficient to show that $|x|^\rho \tilde{u}$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$ and satisfies an estimate having the form of (2.4′).

One sees from (2.6′) that $(m - s)$ is nonnegative. Consider first the case in which $(m - s)$ is positive. Note that, for $|\alpha| = s$, $(\partial^\alpha/\partial x^\alpha)\,\Gamma(x)$ is homogeneous of degree $(m - n - s)$ since, by (2.6′), s is greater than $(m - n)$, (see [6]). Then

$$|\tilde{u}(x)| \leqslant C \int_0^1 \frac{1}{t^{n+s}} \left\{ \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-m+s}} |y|^s |f(y/t)|\, dy \right\} dt$$

or

$$||x|^\rho \tilde{u}(x)| \leqslant C \int_0^1 \frac{1}{t^{n+s}}$$
$$\times \left\{ \int_{\mathbb{R}^n} \frac{1}{|x|^{-\rho} |x - y|^{n-m+s} |y|^{m+\rho-s}} |y|^{m+\rho} |f(y/t)|\, dy \right\} dt$$

for appropriate constants C independent of u. In virtue of (2.1) and (2.6) and the assumption that $(m - s)$ is positive, one may apply Lemma 2.1 with $a = -\rho$ and $b = m + \rho - s$ to obtain

$$\begin{aligned} \| |x|^\rho \tilde{u} \|_p &\leqslant C \int_0^1 \frac{1}{t^{n+s}} \| |y|^{m+\rho} f(y/t) \|_p \, dt \\ &\leqslant C \| |y|^{m+\rho} f(y) \|_p \int_0^1 \frac{t^{m+\rho+n/p}}{t^{n+s}}\, dt \end{aligned}$$

for some constant C independent of u. Since $m + \rho + n/p - n - s > -1$ by (2.6), the integral on the right side of this inequality is finite. Hence, $|x|^\rho \tilde{u}$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$ and the estimate (2.4′) is established for the case in which $(m - s)$ is positive.

The remaining case is that in which s equals m. Again, $(\partial^\alpha/\partial x^\alpha)\,\Gamma(x)$ is homogeneous of degree $-n$ whenever $|\alpha| = m = s$. Then

$$|x|^\rho \tilde{u}(x) = \sum_{|\alpha|=s} \frac{s}{\alpha!} \int_0^1 \frac{(1 - t)^{s-1}}{t^{n+s}}$$
$$\times \left\{ \int_{\mathbb{R}^n} \frac{\Omega_\alpha(x - y)}{|x|^{-\rho} |x - y|^n |y|^\rho} (-y)^\alpha |y|^\rho f(y/t)\, dy \right\} dt,$$

where

$$\Omega_\alpha(x) = |x|^n \frac{\partial^\alpha}{\partial x^\alpha} \Gamma(x) \qquad \text{for } |\alpha| = s.$$

It is well known and easily verified, that the functions Ω_α, for $|\alpha| = s$, satisfy the hypotheses of Lemma 2.2. Furthermore, it follows from (2.6′) that $r = m$ in the present circumstances, and so the conditions (2.1) imply that $\rho < n/p'$. Consequently, Lemma 2.2 may be applied with $a = -\rho$ to obtain the estimate

$$\begin{aligned}\| \, |x|^{\rho} \tilde{u} \|_p &\leqslant C \int_0^1 \frac{1}{t^{n+m}} \| \, |y|^{m+\rho} f(y/t) \|_p \, dt \\ &\leqslant C \| \, |y|^{m+\rho} f(y) \|_p \int_0^1 \frac{t^{m+\rho+n/p}}{t^{n+m}} \, dt,\end{aligned}$$

for an appropriate constant C independent of u. As before, the integral on the right side of this inequality is finite, and so $|x|^{\rho} \tilde{u}$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$ and the estimate (2.4′) is established.

(c) Inequality (2.2) follows easily from the estimate (2.4) and the following general inequality: For any real number ρ, there exists a constant C for which the estimate

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{|\alpha| \leqslant m} \left| \, |x|^{|\alpha|+\rho} \frac{\partial^\alpha}{\partial x^\alpha} u(x) \right|^p dx \\ \leqslant C \int_{\mathbb{R}^n} [| \, |x|^{m+\rho} A_\infty u(x)|^p + | \, |x|^{\rho} u(x)|^p] \, dx\end{aligned} \tag{2.7}$$

holds for all u in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ for which the right side is finite. To establish the inequality (2.7), note that there exists a constant C, independent of R, for which the wellknown interior elliptic estimate

$$R^{p|\alpha|} \int_{R \leqslant |x| \leqslant 2R} \left| \frac{\partial^\alpha}{\partial x^\alpha} u(x) \right|^p dx \leqslant C \int_{R/2 \leqslant |x| \leqslant 3R} [R^{mp} \, | A_\infty u(x)|^p + | u(x)|^p] \, dx \tag{2.8}$$

holds for all u in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ and all α with $|\alpha| \leqslant m$.
(That a single constant C suffices for all positive R is seen by stretching the independent variables.) From this inequality, it immediately follows that, for some constant C,

$$\begin{aligned}\int_{R \leqslant |x| \leqslant 2R} \sum_{|\alpha| \leqslant m} \left| \, |x|^{|\alpha|+\rho} \frac{\partial^\alpha}{\partial x^\alpha} u(x) \right|^p dx \\ \leqslant C \int_{R/2 \leqslant |x| \leqslant 3R} [| \, |x|^{m+\rho} A_\infty u(x)|^p + | \, |x|^{\rho} u(x)|^p] \, dx\end{aligned}$$

for all u in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ and all α with $|\alpha| \leqslant m$.

Letting R take on the values 2^j, $j = \pm 1, \pm 2, \ldots$, in this last inequality and summing over j, one obtains the desired estimate (2.7). This completes the proof of the theorem.

3. A Preliminary Result for Operators with Variable Coefficients

In this section we shall derive the analog of inequality (2.7) for elliptic operators whose coefficients approach those of the operator A at infinity with a certain swiftness. With some effort, it can be shown that the inequality (2.7) is valid as it stands for such elliptic operators. However, since we will be interested primarily in the behavior of functions near infinity, we replace $|x|$ by

$$\sigma(x) = (1 + |x|^2)^{1/2}$$

and prove a similar estimate with less difficulty.

THEOREM 3.1. *Let ρ be any real number, and consider an elliptic partial differential operator*

$$Au(x) = A_\infty u(x) + \sum_{|\alpha| \leq m} b_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

whose coefficients satisfy the following conditions:

(i) *Whenever* $|\alpha| = m$,

$$\limsup_{|x| \to \infty} |b_\alpha(x)| < \delta$$

for some positive δ;

(ii) *Whenever* $|\alpha| < m$,

$$\sup_{x \in \mathbb{R}^n} |x|^{m-|\alpha|} |b_\alpha(x)| < \infty.$$

If δ is sufficiently small, then there is a constant C for which the estimate

$$\sum_{|\alpha| \leq m} \left\| \sigma^{|\alpha|+\rho} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \leq C\{\| \sigma^{m+\rho} Au \|_p + \| \sigma^\rho u \|_p\} \tag{3.1}$$

holds for all u in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ such that $\sigma^{m+\rho}Au$ and $\sigma^\rho u$ are in $L_p(\mathbb{R}^n; \mathbb{C}^k)$.

It is seen in the proof of Theorem 3.1 that the necessary smallness of δ is determined by the size and smoothness of the coefficients of A inside a ball of finite radius about the origin in $\mathbb{R}^n$.

Proof. First note that the operator A may be written as a sum

$$Au(x) = A_0 u(x) + \sum_{|\alpha| \leq m} b_\alpha'(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x), \tag{3.2}$$

where A_0 is an elliptic operator such that $A_0 = A_\infty$ for $|x|$ sufficiently large, the $b_\alpha'(x)$ satisfy condition (ii) of the theorem for $|\alpha| < m$, and

$$\sup_{x \in \mathbb{R}^n} |b_\alpha'(x)| < \delta \qquad \text{for } |\alpha| = m$$

Then there exists a constant C for which the estimate

$$\sum_{|\alpha|\leqslant m}\left\|\sigma^{|\alpha|+\rho}\frac{\partial^\alpha}{\partial x^\alpha}u\right\|_p\leqslant C\{\|\sigma^{m+\rho}A_0u\|_p+\|\sigma^\rho u\|_p\}\tag{3.1'}$$

holds for all u in $H_{m,p}(\mathbb{R}^n;\mathbb{C}^k)$ such that $\sigma^{m+\rho}A_0u$ and $\sigma^\rho u$ are in $L_p(\mathbb{R}^n;\mathbb{C}^k)$. This follows easily from (2.7) and the interior estimates (on a bounded region) for a uniformly elliptic operator analogous to (2.8) (see [2]).

Let $\{\zeta_R\}_{1\leqslant R<\infty}$ be a collection of infinitely differentiable functions satisfying the following conditions

(i) For each $R\geqslant 1$, $0\leqslant\zeta_R(x)\leqslant 1$ for all x in $\mathbb{R}^n$, $\zeta_R(x)=1$ for $|x|\leqslant R$, and $\zeta_R(x)=0$ for $|x|\geqslant 2R$.

(ii) There exists a constant C, independent of R, such that the inequality

$$|\partial^\alpha\zeta_R(x)/\partial x^\alpha|\leqslant C\sigma(x)^{-|\alpha|}\tag{3.3}$$

holds for all x in $\mathbb{R}^n$ and all α with $|\alpha|\leqslant m$.
It follows from (3.3) that, for some constant C independent of R, there is an inequality

$$|\partial^\alpha\zeta_R{}^m(x)/\partial x^\alpha|\leqslant C\sigma(x)^{-|\alpha|}\,\zeta_R(x)^{m-|\alpha|}\tag{3.3'}$$

for all x in $\mathbb{R}^n$ and all α with $|\alpha|\leqslant m$. In the following, the subscript R on each function ζ_R is suppressed whenever there is no danger of misunderstanding.

Let u be an element of $H_{m,p}(\mathbb{R}^n;\mathbb{C}^k)$ such that $\sigma^{m+\rho}Au$ and $\sigma^\rho u$ are in $L_p(\mathbb{R}^n;\mathbb{C}^k)$. Applying (3.1′) to the function $\zeta^m u$, we find

$$\sum_{|\alpha|\leqslant m}\left\|\sigma^{|\alpha|+\rho}\frac{\partial^\alpha}{\partial x^\alpha}(\zeta^m u)\right\|_p\leqslant C\|\sigma^{m+\rho}A_0(\zeta^m u)\|_p+C\|\sigma^\rho\zeta^m u\|_p\,.$$

Using (3.3′), one easily infers that

$$\begin{aligned}\sum_{|\alpha|\leqslant m}&\left\|\sigma^{|\alpha|+\rho}\zeta^m\frac{\partial^\alpha}{\partial x^\alpha}u\right\|_p\\&\leqslant C\|\sigma^{m+\rho}\zeta^m A_0u\|_p+C\sum_{|\beta|<m}\left\|\sigma^{|\beta|+\rho}\zeta^{|\beta|}\frac{\partial^\beta}{\partial x^\beta}u\right\|_p\\&\leqslant C\|\sigma^{m+\rho}\zeta^m Au\|_p+C\sum_{|\alpha|\leqslant m}\left\|\zeta^m\sigma^{m+\rho}b'_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}u\right\|_p\\&\quad+C\sum_{|\beta|<m}\left\|\sigma^{|\beta|+\rho}\zeta^{|\beta|}\frac{\partial^\beta}{\partial x^\beta}u\right\|_p\end{aligned}$$

for appropriate constants C independent of u and R. It follows from the properties of the coefficients b_α' that, if δ is sufficiently small, then the following inequality holds with some constant $\theta < 1$:

$$\sum_{|\alpha| \leqslant m} \left\| \sigma^{|\alpha|+\rho} \zeta^m \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \leqslant C \| \sigma^{m+\rho} A u \|_p + \theta \sum_{|\alpha|=m} \left\| \sigma^{m+\rho} \zeta^m \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p$$
$$+ C \sum_{|\beta| < m} \left\| \sigma^{|\beta|+\rho} \zeta^{|\beta|} \frac{\partial^\beta}{\partial x^\beta} u \right\|_p .$$

Hence,

$$\sum_{|\alpha| \leqslant m} \left\| \sigma^{|\alpha|+\rho} \zeta^m \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \leqslant C \| \sigma^{m+\rho} A u \|_p + C \sum_{|\beta| < m} \left\| \sigma^{|\beta|+\rho} \zeta^{|\beta|} \frac{\partial^\beta}{\partial x^\beta} u \right\|_p$$

with a constant C independent of u and R. Using Lemma 3.1 below, one obtains the inequality

$$\sum_{|\alpha| \leqslant m} \left\| \sigma^{|\alpha|+\rho} \zeta^m \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p$$
$$\leqslant C \| \sigma^{m+\rho} A u \|_p + \frac{1}{2} \sum_{|\alpha|=m} \left\| \sigma^{m+\rho} \zeta^m \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p + C \| \sigma^\rho u \|_p ,$$

from which the desired inequality (3.1) follows on letting $R \to \infty$.

LEMMA 3.1. *Let $\{\zeta_R\}_{1 \leqslant R < \infty}$ be a collection of Lipschitz continuous real-valued functions on $\mathbb{R}^n$ satisfying the following conditions*

(i) *For each $R \geqslant 1$, $0 \leqslant \zeta_R(x) \leqslant 1$ for all x in $\mathbb{R}^n$, $\zeta_R(x) = 1$ for $|x| \leqslant R$, and $\zeta_R(x) = 0$ for $|x| \geqslant 2R$.*

(ii) *There exists a constant C_1, independent of R, for which*

$$| \zeta_R(x) - \zeta_R(y) | \leqslant \frac{C_1}{R} | x - y |$$

for all x and y in $\mathbb{R}^n$.

Let $\sigma(x) = (1 + x^2)^{1/2}$ and let ρ be any real number. Then, for any positive ϵ, there exists a constant $C(\epsilon)$ independent of R for which the inequality

$$\sum_{|\alpha|=k} \left\| \sigma^{k+\rho} \zeta_R^{\,k} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|$$
$$\leqslant \epsilon \sum_{|\alpha|=m} \left\| \sigma^{m+\rho} \zeta_R^{\,m} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p + C(\epsilon) \| \sigma^\rho u \|_p , \qquad 0 \leqslant k < m, \tag{3.4}$$

holds for all u in $H_{m,p}(\mathbb{R}^n)$.

Proof. Again suppressing the subscript R on each function ζ_R we note that it suffices to show that for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ independent of R for which the following inequality holds:

$$\sum_{|\alpha|=k} \left\| \sigma^{k+\rho}\zeta^k \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \leqslant \epsilon \sum_{|\alpha|=k+1} \left\| \sigma^{k+1+\rho}\zeta^{k+1} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p + C(\epsilon) \sum_{|\alpha|=k-1} \left\| \sigma^{k-1+\rho}\zeta^{k-1} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \tag{3.4'}$$

for all $u \in H_{m,p}(\mathbb{R}^n)$ and $0 < k < m$. The desired inequality (3.4) then follows easily.

Our derivation of (3.4') is based on the following well-known result, in which we set

$$| D^k u |^p = \sum_{|\alpha|=k} \left| \frac{\partial^\alpha}{\partial x^\alpha} u(x) \right|^p.$$

There is a constant C' such that for any cube Ω with sides of length s parallel to the coordinate axes, and for any u in $H_{m,p}(\Omega)$ and any $\epsilon' > 0$,

$$\int_\Omega | D^k u |^p \, dx \leqslant \epsilon' \int_\Omega | D^{k+1} u |^p \, dx + C' \left(\frac{1}{\epsilon'} + s^{-p} \right) \int_\Omega | D^{k-1} u |^p \, dx, \qquad 0 < k < m. \tag{3.5}$$

(This result follows from the corresponding result for the unit cube by a stretching.)

Let Ω be any such closed cube in which ζ does not vanish. From (3.5) one has

$$\begin{aligned}
\int_\Omega & | \sigma^{\rho+k}\zeta^k |^p \, | D^k u |^p \, dx \\
&\leqslant \max_\Omega | \sigma^{\rho+k}\zeta^k |^p \cdot \int_\Omega | D^k u |^p \, dx \\
&\leqslant \max_\Omega | \sigma^{\rho+k}\zeta^k |^p \left\{ \epsilon' \int_\Omega | D^{k+1} u |^p \, dx + C' \left(\frac{1}{\epsilon'} + s^{-p} \right) \int_\Omega | D^{k-1} u |^p \, dx \right\} \\
&\leqslant \frac{\max_\Omega | \sigma^{\rho+k}\zeta^k |^p}{\min_\Omega | \sigma^{\rho+k}\zeta^k |^p} \left\{ \frac{\epsilon'}{\min_\Omega | \sigma\zeta |^p} \int_\Omega | \sigma^{\rho+k+1}\zeta^{k+1} |^p \, | D^{k+1} u |^p \, dx \right. \\
&\quad \left. + C' \left(\frac{1}{\epsilon'} + s^{-p} \right) [\min_\Omega | \sigma\zeta |^p] \int_\Omega | \sigma^{\rho+k-1}\zeta^{k-1} |^p \, | D^{k-1} u |^p \, dx \right\}.
\end{aligned} \tag{3.6}$$

We shall make use of the following (essentially special) case of Whitney's Lemma. For completeness, the proof is given at the end of this section.

LEMMA 3.2. *Let $f(x)$ be a bounded nonnegative continuous function in $\mathbb{R}^n$ satisfying for some $K > 0$*

$$|f(x) - f(y)| \leqslant K\,|x - y| \qquad \textit{for } x, y \in R^n.$$

There is a covering of the subset of $\mathbb{R}^n$ on which f is positive by closed cubes with faces parallel to the coordinate hyperplanes and with nonoverlapping interiors, $\{Q_j\}$, $j = 1, 2,\ldots,$ such that

$$s_j = \text{side length of } Q_j \leqslant \min_{Q_j} f(x) \leqslant \max_{Q_j} f(x) \leqslant 2s_j(1 + Kn^{1/2}).$$

We shall apply Lemma 3.2 to the function $f = \sigma\zeta$ which satisfies the conditions of the lemma with $K = 1 + 2C_1$. In the resulting covering by cubes Q_j, $j = 1, 2,\ldots,$ we see from the Lipschitz condition of ζ that

$$\max_{Q_j} \zeta \leqslant \min_{Q_j} \zeta + \frac{C_1}{R}\, n^{1/2}\, s_j \leqslant \min_{Q_j} \zeta \left(1 + \frac{C_1}{R}\, n^{1/2} \max_{Q_j} \sigma\right).$$

Since each Q_j is contained in the ball $|x| \leqslant 2R$ we see that

$$\max_{Q_j} \zeta / \min_{Q_j} \zeta \leqslant C$$

where C is a constant independent of R and j. Furthermore, we have

$$\max_{Q_j} \sigma \leqslant \min_{Q_j} \sigma + n^{1/2}\, s_j \leqslant \min_{Q_j} \sigma(1 + n^{1/2}).$$

Consequently, we may assert that for a constant C independent of R and of j,

$$\max_{Q_j} |\sigma^{\rho+k}\zeta^k|^p / \min_{Q_j} |\sigma^{\rho+k}\zeta^k|^p \leqslant C.$$

If we use this inequality in (3.6) with $\Omega = Q_j$ we may infer that

$$\begin{aligned}
&\int_{Q_j} |\sigma^{\rho+k}\zeta^k|^p\, |D^k u|^p\, dx \\
&\qquad \leqslant C \left\{ (\epsilon' / \min_{Q_j} |\sigma\zeta|^p) \int_{Q_j} |\sigma^{\rho+k+1}\zeta^{k+1}|^p\, |D^{k+1}u|^p\, dx \right. \\
&\qquad\qquad \left. + C' \left(\frac{1}{\epsilon'} + s_j^{-p}\right) [\min_{Q_j} |\sigma\zeta|^p] \int_{Q_j} |\sigma^{\rho+k-1}\zeta^{k-1}|^p\, |D^{k-1}u|^p\, dx \right\}.
\end{aligned}$$

Setting $\epsilon' = \epsilon / C \min_{Q_j} | \sigma\zeta |^p$, and using the last inequality of Lemma 3.2 for $f = \sigma\zeta$ and $K = 1 + 2C_1$, we find on summing that, for any $\epsilon > 0$,

$$\sum_j \int_{Q_j} | \sigma^{\rho+k}\zeta^k |^p \, | D^k u |^p \, dx$$

$$\leqslant \epsilon \sum_j \int_{Q_j} | \sigma^{\rho+k+1}\zeta^{k+1} |^p \, | D^{k+1} u |^p \, dx$$

$$+ CC' \left(\frac{C}{\epsilon} + 2^p(1 + Kn^{1/2})^p \right) \sum_j \int_{Q_j} | \sigma^{\rho+k-1}\zeta^{k-1} |^p \, | D^{k-1} u |^p \, dx.$$

This is equivalent to (3.4′), and Lemma 3.1 is proved.

Proof of Lemma 3.2. Let $M = \sup f$, and let $z_M{}^n$ be the set of points in $\mathbb{R}^n$ with coordinates which are integral multiples of M. Let L be the collection of closed cubes with side M centered at the points of $z_M{}^n$ and with edges parallel to the axes. Denote by K_0 the set of those cubes in L on which $\min f(x) \geqslant M$. Subdivide each of the remaining cubes into 2^n cubes of side length $M/2$ and denote by K_1 the set of the latter with the property:

$$\min_{\text{cube}} f \geqslant M/2.$$

Continuing this process we obtain a collection Q of cubes all of which have the property

$$\text{min of } f \text{ on the cube} \geqslant \text{side length of the cube.}$$

We show first that the set where f is positive is covered by cubes of Q. For example suppose x does not belong to any of the cubes of Q. There is, then, a sequence of cubes Q^i, $i = 1, 2, \ldots$, containing x, and with

$$\min_{Q^i} f < M\, 2^{-i} = \text{side length of } Q^i.$$

Because of the Lipschitz conditions of f it follows that

$$f(x) \leqslant M\, 2^{-i}(1 + n^{1/2}K)$$

and hence $f(x) = 0$.

Now order the cubes in Q into a sequence Q_j with side lengths s_j, $j = 1, 2, \ldots$. From our construction we see that if $x \in Q_j$, then

$$f(x) \leqslant 2s_j + 2n^{1/2} s_j K$$

and the lemma is proved.

4. Null Spaces of Elliptic Operators in $\mathbb{R}^n$

As before let A_∞ denote a given linear elliptic partial differential operator with constant coefficients which is homogeneous of order m and which acts on $\mathbb{C}^k$-valued functions of n independent variables. We wish to investigate the nullspaces of linear elliptic partial differential operators of order m whose coefficients approach those of A_∞ at infinity at a certain rate. Of specific concern is the dimension of the null space $N_p(A)$ of such an operator A in each Banach space $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$, $1 < p < \infty$, and the behavior of this dimension when the operator is perturbed slightly in a prescribed manner. The situation depends slightly on whether or not $(m - n/p')$ is a nonnegative integer.

THEOREM 4.1. *Let p and p' satisfy $1 < p < \infty$ and $1/p + 1/p' = 1$. Consider an elliptic partial differential operator*

$$Au(x) = A_\infty u(x) + \sum_{|\alpha| \leqslant m} b_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

whose coefficients satisfy the following conditions: There exists a positive number δ such that

$$\limsup_{|x| \to \infty} |x|^{m-|\alpha|} |b_\alpha(x)| < \delta$$

for each α with $|\alpha| \leqslant m$; if δ is sufficiently small, then the dimension of $N_p(A)$ is finite.

Proof. Consider first the case in which $m - n/p'$ is not a nonnegative integer. $N_p(A)$ is a closed subspace of $L_p(\mathbb{R}^n; \mathbb{C}^k)$ and is finite-dimensional if and only if the set $S = \{u \in N_p(A): \| u \|_p = 1\}$ is compact. Therefore, to prove the theorem, it suffices to show that, for sufficiently small δ, every sequence in the closed subset S of $L_p(\mathbb{R}^n; \mathbb{C}^k)$ contains a subsequence which is Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$.

For some R sufficiently large that

$$\sup_{|x| \geqslant R} |x|^{m-|\alpha|} |b_\alpha(x)| < \delta$$

for each α with $|\alpha| \leqslant m$, let ϕ be an infinitely differentiable real-valued function on $\mathbb{R}^n$ with compact support which is such that $\phi(x) = 1$ whenever $|x| \leqslant R$. Suppose that an arbitrary sequence $\{u_i\}$ in S is given. If δ is sufficiently small, or if the coefficients $b_\alpha(x)$ are uniformly continuous on $\mathbb{R}^n$ whenever $|\alpha| = m$, then there exists a constant C for which the estimate

$$\| u \|_{m,p} \leqslant C\{\| Au \|_p + \| u \|_p\} \tag{4.1}$$

holds for all u in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ (see [2]). Hence the sequence $\{u_i\}$ is bounded in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$. In light of the Rellich compactness theorem (see [3, p. 31]), then, it may be assumed that the sequence $\{\phi u_i\}$ is Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. Therefore, the theorem will be proved if it can be shown that, for δ sufficiently small, there is a subsequence of $\{(1-\phi)\, u_i\}$ which is Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$.

Assume that δ is so small that an estimate of the form (3.1) holds with $\rho = 0$. It follows from this estimate that

$$\sum_{|\alpha|\leqslant m} \left\| \, |x|^{|\alpha|} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p$$

is finite for any u in $N_p(A)$. This, in turn, implies that the right side of the inequality (2.2) with $\rho = 0$ is finite for any u in $N_p(A)$, for if u is in $N_p(A)$, then

$$\| \, |x|^m A_\infty u \|_p = \| \, |x|^m (A_\infty - A)\, u \|_p \leqslant C \sum_{|\alpha|\leqslant m} \left\| \, |x|^{|\alpha|} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p$$

for an appropriate constant C. From this, it is seen that $|x|^m A_\infty(1-\phi)\, u$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$ whenever u is in $N_p(A)$, and so the estimate (2.2) may be applied with $\rho = 0$ to $(1-\phi)\, u$ for such u. Then for any u in $N_p(A)$,

$$\begin{aligned}
&\sum_{|\alpha|\leqslant m} \left\| \, |x|^{|\alpha|} (1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \\
&\quad\leqslant \sum_{|\alpha|\leqslant m} \left\| \, |x|^{|\alpha|} \frac{\partial^\alpha}{\partial x^\alpha} (1-\phi)\, u \right\|_p \\
&\qquad + \sum_{|\alpha|\leqslant m} \left\| \, |x|^{|\alpha|} \left[(1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u - \frac{\partial^\alpha}{\partial x^\alpha} (1-\phi)\, u \right] \right\|_p \\
&\quad\leqslant C_0 \| \, |x|^m A_\infty(1-\phi)\, u \|_p \\
&\qquad + \sum_{|\alpha|\leqslant m} \left\| \, |x|^{|\alpha|} \left[(1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u - \frac{\partial^\alpha}{\partial x^\alpha} (1-\phi)\, u \right] \right\|_p \\
&\quad\leqslant C_0 \| \, |x|^m (1-\phi)(A_\infty - A)\, u \|_p \\
&\qquad + C_0 \| \, |x|^m [A_\infty(1-\phi)\, u - (1-\phi)\, A_\infty u] \|_p \\
&\qquad + \sum_{|\alpha|\leqslant m} \left\| \, |x|^{|\alpha|} \left[(1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u - \frac{\partial^\alpha}{\partial x^\alpha} (1-\phi)\, u \right] \right\|_p \\
&\quad\leqslant C_0 \delta \sum_{|\alpha|\leqslant m} \left\| \, |x|^{|\alpha|} (1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \\
&\qquad + C_0 \| \, |x|^m [A_\infty(1-\phi)\, u - (1-\phi)\, A_\infty u] \|_p \\
&\qquad + \sum_{|\alpha|\leqslant m} \left\| \, |x|^{|\alpha|} \left[(1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u - \frac{\partial^\alpha}{\partial x^\alpha} (1-\phi)\, u \right] \right\|_p .
\end{aligned}$$

Consequently, if it is assumed that δ is less than $1/C_0$, then there exists a constant C for which the estimate

$$\sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} (1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p$$

$$\leqslant C \Big\{ \| |x|^m [A_\infty(1-\phi) u - (1-\phi) A_\infty u] \|_p \tag{4.2}$$

$$+ \sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} \left[(1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u - \frac{\partial^\alpha}{\partial x^\alpha} (1-\phi) u \right] \right\|_p \Big\}$$

holds for all u in $N_p(A)$.

Now the sequences

$$\{|x|^m [A_\infty(1-\phi) u_i - (1-\phi) A_\infty u_i]\}$$

and

$$\left\{ |x|^{|\alpha|} \left[(1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u_i - \frac{\partial^\alpha}{\partial x^\alpha} (1-\phi) u_i \right] \right\},$$

for $|\alpha| \leqslant m$, are bounded in $H_{1,p}(\mathbb{R}^n; \mathbb{C}^k)$ and consist of functions whose supports are contained in the support of ϕ. Therefore, it is a consequence of the Rellich Compactness Theorem that there exists a subsequence $\{u_{i_j}\}$ of $\{u_i\}$ such that the sequences

$$\{|x|^m [A_\infty(1-\phi) u_{i_j} - (1-\phi) A_\infty u_{i_j}]\}$$

and

$$\left\{ |x|^{|\alpha|} \left[(1-\phi) \frac{\partial^\alpha}{\partial x^\alpha} u_{i_j} - \frac{\partial^\alpha}{\partial x^\alpha} (1-\phi) u_{i_j} \right] \right\},$$

for $|\alpha| \leqslant m$, are Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. Substituting $u = u_{i_j} - u_{i_l}$ in the inequality (4.2), one verifies immediately that the sequence $\{(1-\phi) u_{i_j}\}$ is itself Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. This completes the proof of the theorem in case $m - n/p'$ is not a nonnegative integer.

Suppose now that $m - n/p'$ is a nonnegative integer. To prove the theorem in this case we show simply that $N_p(A)$ is contained in $N_{\tilde{p}}(A)$ for any $\tilde{p} > p$. Choosing $\tilde{p}$ larger but close to p, and so that $m - n/\tilde{p}'$ is not a nonnegative integer, and taking δ small enough (appropriate for $\tilde{p}$) we may conclude that $\dim N_p(A) \leqslant \dim N_{\tilde{p}}(A) < \infty$. To prove $N_p(A) \subseteq N_{\tilde{p}}(A)$ we make use of the results of [1, Appendix 5]. Let B_1, B_2, and B_3 be concentric balls of radii 1, 2, and 3. According to those results, if $u \in H_{m,p}$ in B_3 and satisfies $Au = 0$ then u is bounded in B_2 and its derivatives up to order m belong to $L_{\tilde{p}}$ in B_1 for any $\tilde{p} > p$. In fact

$$\sup_{B_2} | u |^p \leqslant C \int_{B_3} \sum_{|\alpha| \leqslant m} \left| \frac{\partial^\alpha}{\partial x^\alpha} u \right|^p dx$$

and

$$\int_{B_1} \sum_{|\alpha| \leqslant m} \left| \frac{\partial^\alpha}{\partial x^\alpha} u \right|^{\tilde{p}} \leqslant C \int_{B_2} | u |^{\tilde{p}} dx,$$

where the constant C is independent of the center of the balls. If $u \in H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ we conclude that $| u |$ is bounded in $\mathbb{R}^n$ by some constant M, and from the second inequality we infer that

$$\int_{B_1} \sum_{|\alpha| \leqslant m} \left| \frac{\partial^\alpha}{\partial x^\alpha} u \right|^{\tilde{p}} \leqslant CM^{\tilde{p}-p} \int_{B_2} | u |^p dx.$$

Covering $\mathbb{R}^n$ by such balls B_1 with at most finite intersection and summing, we find

$$\| u \|_{m,\tilde{p}} \leqslant \text{const } M^{\tilde{p}-p} \| u \|_p$$

and hence $u \in N_{\tilde{p}}(A)$.

We now take up the question of the variation of the dimension of the null space $N_p(A)$ under small perturbations of A. The most extensive results are obtained when $m - n/p'$ is not a nonnegative integer.

THEOREM 4.2. *Let p and p' satisfy $1 < p < \infty$ and $1/p + 1/p' = 1$, and assume that $(m - n/p')$ is not a nonnegative integer. Consider an elliptic partial differential operator*

$$Au(x) = A_\infty u(x) + \sum_{|\alpha| \leqslant m} b_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

whose coefficients are such that

$$\limsup_{|x| \to \infty} | x |^{m-|\alpha|} | b_\alpha(x) | < \delta$$

for each α with $| \alpha | \leqslant m$, where δ is some positive number sufficiently small that Theorem 4.1 guarantees the finitedimensionality of $N_p(A)$. Then there exists a positive number ϵ such that if

$$A'u(x) = A_\infty u(x) + \sum_{|\alpha| \leqslant m} b_\alpha{}'(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

is an elliptic partial differential operator whose coefficients satisfy

$$\sup_{x \in \mathbb{R}^n}(1 + | x |)^{m-|\alpha|} | b_\alpha(x) - b_\alpha'(x)| < \epsilon$$

for each α *with* $| \alpha | \leqslant m$, *then the dimension of* $N_p(A')$ *is less than or equal to the dimension of* $N_p(A)$.

Proof. Denote the dimension of $N_p(A)$ by q and suppose that the theorem is false. Then for each positive integer i, there exists an elliptic operator

$$A_i u(x) = A_x u(x) + \sum_{|\alpha| \leqslant m} b_\alpha{}^i(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

having the property that

$$\sup_{x \in \mathbb{R}^n}(1 + | x |)^{m-|\alpha|} | b_\alpha(x) - b_\alpha{}^i(x)| < i^{-1}$$

for each α with $| \alpha | \leqslant m$, and such that the dimension of $N_p(A_i)$ is greater than q. Observe that a positive number ϵ_0 may be found such that any subspace of $L_p(\mathbb{R}^n; \mathbb{C}^k)$ with dimension greater than q contains an element of norm one whose distance from $N_p(A)$ is at least ϵ_0. In particular, an element u_i of $N_p(A_i)$ may be chosen for each i such that $\| u_i \|_p = 1$ and such that the distance of u_i from $N_p(A)$ is greater than ϵ_0. Now it must be the case that the sequence $\{Au_i\}$ converges to zero in $L_p(\mathbb{R}^n; \mathbb{C}^k)$; therefore, if a subsequence of $\{u_i\}$ exists which is Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$, it follows from the closedness of A that the elements of this subsequence converge to a function in $N_p(A)$. But no subsequence of $\{u_i\}$ can converge to an element of $N_p(A)$ since the functions u_i all lie at a distance greater than ϵ_0 from $N_p(A)$. To prove the theorem, then, it suffices to show that there exists a subsequence of $\{u_i\}$ which is Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$.

Now it is implicit in the assumption concerning the size of δ not only that δ is less than the inverse of the constant C_0 appearing in the estimate (2.2) but also that there exist estimates involving the given operator A which have the forms of (4.1) and (3.1) with $\rho = 0$. It follows from these estimates and from the "nearness" of the operators A_i to A that, for sufficiently large i, there exist analogous estimates for the operators A_i in which the constants that appear may be chosen independent of i. Note for later reference that, consequently, the sequence $\{u_i\}$ is bounded in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$, the sequence $\{| x |^m Au_i\}$ converges to zero in $L_p(\mathbb{R}^n; \mathbb{C}^k)$, and the quantity

$$\sum_{|\alpha| \leqslant m} \left\| | x |^{|\alpha|} \frac{\partial^\alpha}{\partial x^\alpha} u_i \right\|_p$$

is finite for sufficiently large i.

For some R sufficiently large that $\sup_{|x|\geqslant R} |x|^{m-|\alpha|} |b_\alpha(x)|$ is less than δ for each α with $|\alpha| \leqslant m$, let ϕ be an infinitely differentiable real-valued function on $\mathbb{R}^n$ with compact support which is such that $\phi(x) = 1$ whenever $|x| \leqslant R$. Since $\{u_i\}$ is bounded in $H_{m,p}(\mathbb{R}^k; \mathbb{C}^k)$, it may be assumed in light of the Rellich Compactness Theorem that the sequence $\{\phi u_i\}$ is Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. Then it only remains to find a subsequence of $\{(1-\phi)u_i\}$ which is Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. As in the proof of the preceding theorem, it is easily seen that $|x|^m A_\infty(1-\phi)(u_i - u_j)$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$ for sufficiently large i and j, and one may apply the estimate (2.2) to obtain

$$\begin{aligned}
&\sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} (1-\phi) \frac{\partial^\alpha}{\partial x^\alpha}(u_i - u_j) \right\|_p \\
&\quad\leqslant \sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} \frac{\partial^\alpha}{\partial x^\alpha}(1-\phi)(u_i - u_j) \right\|_p \\
&\qquad + \sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} \left[(1-\phi)\frac{\partial^\alpha}{\partial x^\alpha}(u_i - u_j) - \frac{\partial^\alpha}{\partial x^\alpha}(1-\phi)(u_i-u_j)\right] \right\|_p \\
&\quad\leqslant C_0 \| |x|^m A_\infty(1-\phi)(u_i-u_j)\|_p \\
&\qquad + \sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} \left[(1-\phi)\frac{\partial^\alpha}{\partial x^\alpha}(u_i - u_j) - \frac{\partial^\alpha}{\partial x^\alpha}(1-\phi)(u_i-u_j)\right] \right\|_p \\
&\quad\leqslant C_0 \| |x|^m (1-\phi)(A_\infty - A)(u_i-u_j)\|_p \\
&\qquad + C_0 \| |x|^m (1-\phi) A(u_i-u_j)\|_p \\
&\qquad + C_0 \| |x|^m [A_\infty(1-\phi)(u_i-u_j) - (1-\phi)A_\infty(u_i-u_j)]\|_p \\
&\qquad + \sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} \left[(1-\phi)\frac{\partial^\alpha}{\partial x^\alpha}(u_i - u_j) - \frac{\partial^\alpha}{\partial x^\alpha}(1-\phi)(u_i-u_j)\right] \right\|_p .
\end{aligned}$$

Since

$$\begin{aligned}
C_0 \| |x|^m (1-\phi)(A_\infty - A)(u_i-u_j)\|_p \\
\leqslant C_0\delta \sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} (1-\phi) \frac{\partial^\alpha}{\partial x^\alpha}(u_i - u_j) \right\|_p
\end{aligned}$$

and since $C_0\delta$ is less than one, it follows that there exists a constant C for which the inequality

$$\begin{aligned}
&\sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} (1-\phi) \frac{\partial^\alpha}{\partial x^\alpha}(u_i - u_j) \right\|_p \\
&\quad\leqslant C \Big\{ \| |x|^m (1-\phi) A(u_i-u_j)\|_p \\
&\qquad + \| |x|^m [A_\infty(1-\phi)(u_i-u_j) - (1-\phi)A_\infty(u_i-u_j)]\|_p \\
&\qquad + \sum_{|\alpha|\leqslant m} \left\| |x|^{|\alpha|} \left[(1-\phi)\frac{\partial^\alpha}{\partial x^\alpha}(u_i - u_j) - \frac{\partial^\alpha}{\partial x^\alpha}(1-\phi)(u_i-u_j)\right] \right\|_p \Big\}
\end{aligned} \tag{4.3}$$

holds for all sufficiently large i and j. Note that the sequences

$$\{| x |^m [A_\infty(1 - \phi) u_i - (1 - \phi) A_x u_i]\}$$

and

$$\left\{| x |^{|\alpha|} \left[(1 - \phi) \frac{\partial^\alpha}{\partial x^\alpha} u_i - \frac{\partial^\alpha}{\partial x^\alpha} (1 - \phi) u_i\right]\right\},$$

for $|\alpha| \leqslant m$, are bounded in $H_{1,p}(\mathbb{R}^n; \mathbb{C}^k)$ and consist of functions whose supports are contained in the support of ϕ. Then it follows from the Rellich Compactness Theorem that there exists a subsequence $\{u_{i_j}\}$ of $\{u_i\}$ which is such that the sequence

$$\{| x |^m [A_\infty(1 - \phi) u_{i_j} - (1 - \phi) A_\infty u_{i_j}]\}$$

and

$$\left\{| x |^{|\alpha|} \left[(1 - \phi) \frac{\partial^\alpha}{\partial x^\alpha} u_{i_j} - \frac{\partial^\alpha}{\partial x^\alpha} (1 - \phi) u_{i_j}\right]\right\},$$

for $|\alpha| \leqslant m$, are Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. Furthermore, the sequence $\{| x |^m (1 - \phi) A u_i\}$ converges to zero in $L_p(\mathbb{R}^n; \mathbb{C}^k)$, and so a particular consequence of inequality (4.3) above is that the sequence $\{(1 - \phi) u_{i_j}\}$ is itself Cauchy in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. This completes the proof of the theorem.

If $m - n/p'$ is a nonnegative integer, then the conclusion of Theorem 4.2 is false. This is demonstrated by the following example, in which $m = 1$ and $n = p = p' = 2$.

Example. Let A_∞ denote the Cauchy–Riemann operator on $H_1(\mathbb{R}^2; \mathbb{C}^2)$, i.e.,

$$A_\infty U(x) = \frac{\partial}{\partial x_1} U(x) + M \frac{\partial}{\partial x_2} U(x),$$

where

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for U in $H_1(\mathbb{R}^2; \mathbb{R}^2)$. Let ζ be an infinitely differentiable real-valued function on $\mathbb{R}^n$ which vanishes in a neighborhood of the origin and which satisfies $\zeta(x) = 1$ for $| x | \geqslant 1$. For $\epsilon \geqslant 0$, and x in $\mathbb{R}^2$, define

$$\begin{pmatrix} u_{1\epsilon} \\ u_{2\epsilon} \end{pmatrix} = U_\epsilon(x) = \frac{\zeta(x)}{(1 + | x |^2)^\epsilon} \begin{pmatrix} \mathrm{Re}(1/[x_1 + ix_2]) \\ \mathrm{Im}(1/[x_1 + ix_2]) \end{pmatrix} + (1 - \zeta(x)) \begin{pmatrix} i \\ 0 \end{pmatrix},$$

and set

$$T_\epsilon = \begin{pmatrix} u_{1\epsilon} & -\overline{u_{2\epsilon}} \\ u_{2\epsilon} & \overline{u_{1\epsilon}} \end{pmatrix}.$$

Since $U_\epsilon(x)$ never vanishes we have

$$T_\epsilon^{-1} = (|\, u_{1\epsilon}\,|^2 + |\, u_{2\epsilon}\,|^2)^{-1} \begin{pmatrix} \overline{u_{1\epsilon}} & \overline{u_{2\epsilon}} \\ -u_{2\epsilon} & u_{1\epsilon} \end{pmatrix}, \qquad \text{and} \qquad T_\epsilon^{-1} U_\epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus U_ϵ satisfies the system of equations

$$A_\infty(T_\epsilon^{-1} U_\epsilon) = 0 \tag{4.4}$$

or

$$A_\epsilon U_\epsilon \equiv \frac{\partial}{\partial x_1} U_\epsilon + T_\epsilon M T_\epsilon^{-1} \frac{\partial}{\partial x_2} U_\epsilon + B_\epsilon U_\epsilon = 0, \tag{4.4$'$}$$

where B_ϵ is the matrix

$$B_\epsilon = T_\epsilon \left(\frac{\partial}{\partial x_1} T_\epsilon^{-1} + M \frac{\partial}{\partial x_2} T_\epsilon^{-1}\right).$$

For $\epsilon > 0$, U_ϵ is in $H_1(\mathbb{R}^2; \mathbb{C}^2)$ and thus the dimension of $N_2(A_\epsilon)$ is positive for $\epsilon > 0$. (In fact, the dimension of $N_2(A_\epsilon)$ equals two for $\epsilon > 0$.)

Now for $|\, x\,| \geqslant 1$, the functions $u_{1\epsilon}$, $u_{2\epsilon}$ are real. Therefore,

$$u_{1\epsilon} = \frac{1}{(1 + |\, x\,|^2)^\epsilon} \operatorname{Re} \left(\frac{1}{x_1 + ix_2}\right), \qquad u_{2\epsilon} = \frac{1}{(1 + |\, x\,|^2)^\epsilon} \operatorname{Im} \left(\frac{1}{x_1 + ix_2}\right),$$

and

$$T_\epsilon = u_{1\epsilon} I + u_{2\epsilon} M, \qquad T_\epsilon^{-1} = \frac{(u_{1\epsilon} I - u_{2\epsilon} M)}{u_{1\epsilon}^2 + u_{2\epsilon}^2}.$$

Since $M^2 = -I$ we see that

$$T_\epsilon M T_\epsilon^{-1} = M \qquad \text{for } |\, x\,| \geqslant 1.$$

Furthermore, for $|\, x\,| \geqslant 1$,

$$\begin{aligned} B_\epsilon &= -\left[\left(\frac{\partial}{\partial x_1} T_\epsilon\right) + T_\epsilon M T_\epsilon^{-1} \left(\frac{\partial}{\partial x_2} T_\epsilon\right)\right] \cdot T_\epsilon^{-1} \\ &= -\left[\frac{\partial}{\partial x_1} T_\epsilon + M \frac{\partial}{\partial x_2} T_\epsilon\right] \cdot T_\epsilon^{-1}. \end{aligned}$$

A straightforward calculation yields

$$B_\epsilon = \frac{2\epsilon}{(1+|x|^2)^{1+\epsilon}} T_\epsilon^{-1} = \frac{2\epsilon |x|^2}{(1+|x|^2)} \left(\mathrm{Re}\left(\frac{1}{x_1+ix_2}\right) I - \mathrm{Im}\left(\frac{1}{x_1+ix_2}\right) M\right)$$
$$= \frac{2\epsilon}{1+|x|^2} \begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix}.$$

Hence for $|x| \geqslant 1$ the system (4.4′) takes the form

$$A_\infty U_\epsilon + \frac{2\epsilon}{1+|x|^2} \begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} U_\epsilon = 0.$$

We see also that $B_0(x)$ vanishes for $|x| \geqslant 1$ and, for some constant C independent of ϵ,

$$\sup_{x\in\mathbb{R}^2}(1+|x|)\,|B_\epsilon(x) - B_0(x)| \leqslant \epsilon C.$$

If the conclusion of Theorem 4.2 were valid in this case, it would follow that $\dim N_2(A_0) > 0$.

We claim, however, that $\dim N_2(A_0) = 0$, i.e., the only solution U in $H_1(\mathbb{R}^2; \mathbb{C}^2)$ of $A_0 U = 0$ is $U = 0$. For if U is a solution then, according to (4.4),

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T_0^{-1} U$$

satisfies

$$A_\infty V = 0.$$

If follows that $f = \mathrm{Re}\, v_1 + i\, \mathrm{Re}\, v_2$ and $g = \mathrm{Im}\, v_1 + i\, \mathrm{Im}\, v_2$ are holomorphic functions of $z = x_1 + ix_2$. Furthermore, since $U = T_0 v$ is in $H_1(\mathbb{R}^2; \mathbb{C}^2)$ and, for $|z| \geqslant 1$,

$$T_0 = \begin{pmatrix} \mathrm{Re}(1/z) & -\mathrm{Im}(1/z) \\ \mathrm{Im}(1/z) & \mathrm{Re}(1/z) \end{pmatrix},$$

so that

$$U = \begin{pmatrix} \mathrm{Re}(f/z) & +i\,\mathrm{Re}(g/z) \\ \mathrm{Im}(f/z) & +i\,\mathrm{Im}(g/z) \end{pmatrix},$$

we see that

$$\iint_{|z|\geqslant 1} (|f(z)|^2 + |g(z)|^2)\,|z|^{-2}\, dx_1\, dx_2 < \infty.$$

By Liouville's Theorem, f and g are zero. Hence, U is the trivial solution.

In fact, if no more is required of the operator

$$A = A_\infty + \sum_{|\alpha| \leqslant m} b_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$$

than that $\limsup_{|x| \to \infty} | x |^{m-|\alpha|} | b_\alpha(x)|$ be small for $| \alpha | \leqslant m$, then it appears that the techniques of the proof of Theorem 4.2 cannot be used to obtain an upper-semicontinuity theorem at all when $m - n/p'$ is a nonnegative integer. The reason is that the estimate (2.2) can only be profitably applied in this case with ρ small and positive, and one can easily construct operators A, with $\limsup_{|x| \to \infty} | x |^{m-|\alpha|} | b_\alpha(x)|$ arbitrarily small for $| \alpha | \leqslant m$, which contain elements in their null spaces for which the left side of this estimate is infinite for any positive ρ. However, if more stringent conditions are placed on the coefficients of the operator A, then one obtains the following simplified theory, which is valid for all positive integers m and n and all p with $1 < p < \infty$.

THEOREM 4.3. *Let $\rho \in (0, 1]$ be such that an estimate of the form* (2.2) *holds for all u in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ such that $| x |^{m+\rho} A_\infty u$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$. Consider an elliptic partial differential operator*

$$Au(x) = A_\infty u(x) + \sum_{|\alpha| \leqslant m} b_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

whose coefficients are such that

$$\sup_{x \in \mathbb{R}^n} | x |^{m-|\alpha|+\rho} | b_\alpha(x)| < \infty, \qquad | \alpha | \leqslant m.$$

Then there exists a constant C for which the estimate

$$\sum_{|\alpha| \leqslant m} \left\| | x |^{|\alpha|+\rho} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \leqslant C\{\| | x |^{m+\rho} Au \|_p + \| \sigma^m Au \|_p + \| u \|_p\}$$

holds for all u in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ such that $| x |^{m+\rho} Au$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$, where $\sigma(x) = (1 + | x |^2)^{1/2}$.

Proof. Note that the operator A satisfies the hypotheses of Theorem 3.1. Then, for a function u in $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ such that $| x |^{m+\rho} Au$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$, one has

$$\begin{aligned}
\| | x |^{m+\rho} A_\infty u \|_p &\leqslant \| | x |^{m+\rho} Au \|_p + \left\| | x |^{m+\rho} \sum_{|\alpha| \leqslant m} b_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \\
&\leqslant \| | x |^{m+\rho} Au \|_p + C \sum_{|\alpha| \leqslant m} \left\| | x |^{|\alpha|} \frac{\partial^\alpha}{\partial x^\alpha} u \right\|_p \\
&\leqslant \| | x |^{m+\rho} Au \|_p + C \| \sigma^m Au \|_p + C \| u \|_p
\end{aligned}$$

for appropriate constants C independent of u. Consequently, $|x|^{m+\rho} A_\infty u$ is in $L_p(\mathbb{R}^n; \mathbb{C}^k)$ for such a function u, and the desired estimate follows from the inequality (2.2).

COROLLARY 1. *If ρ and A satisfy the hypotheses of the theorem, then the dimension of $N_p(A)$ is finite.*

COROLLARY 2. *If ρ and A satisfy the hypotheses of the theorem, then there exists a positive number ϵ such that, if*

$$A'u(x) = A_\infty u(x) + \sum_{|\alpha| \leqslant m} b_\alpha'(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

is an elliptic partial differential operator whose coefficients satisfy

$$\sup_{x \in \mathbb{R}^n} (1 + |x|)^{m-|\alpha|+\rho} |b_\alpha - b_\alpha'(x)| < \epsilon \qquad \text{for } |\alpha| \leqslant m,$$

then the dimension of $N_p(A')$ is less than or equal to the dimension of $N_p(A)$.

Corollary 1 follows from Theorem 4.3 with a straightforward application of the Rellich compactness theorem. The proof of Corollary 2 follows closely that of Theorem 4.2 in spirit.

5. An Operator with an Infinite-Dimensional Null Space

A method will now be described for constructing an elliptic partial differential operator in $\mathbb{R}^n$ whose nullspace in any of the Banach spaces $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$, for $1 \leqslant p < \infty$, is infinite-dimensional. The operator produced by the construction very nearly satisfies the hypotheses of Theorem 4.1 in the following sense: If the operator is denoted by

$$A_\infty u(x) + \sum_{|\alpha| \leqslant m} b_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x),$$

where, as before, A_∞ is an elliptic operator with constant coefficients which is homogeneous of order m, then the coefficients are such that

$$\limsup_{|x| \to \infty} |x|^{m-|\alpha|} |b_\alpha(x)|$$

is finite for each α with $|\alpha| \leqslant m$. Therefore, it must be the case that the operator fails to lie within the scope of Theorem 4.1 because the quantities

$$\limsup_{|x| \to \infty} |x|^{m-|\alpha|} |b_\alpha(x)|, \qquad \text{for } |\alpha| \leqslant m,$$

are too large. This seems to indicate that, in order to obtain results which are more extensive than those described in this paper, one must consider properties of the operators at hand in addition to the size of their coefficients near infinity.

The construction begins with an elliptic partial differential operator

$$A_0 u(x) = A_\infty u(x) + \sum_{|\alpha| \leqslant m} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

which satisfies the following conditions

(i) A_∞ is an elliptic partial differential operator with constant coefficients which is homogeneous of order m;

(ii) the coefficients $a_\alpha(x)$, for $|\alpha| \leqslant m$, are infinitely differentiable and have support in the unit ball in $\mathbb{R}^n$;

(iii) there exists a (necessarily infinitely differentiable) nonzero function u_0 having support in the unit ball in $\mathbb{R}^n$ which satisfies $A_0 u_0 = 0$.

An operator satisfying these conditions has been constructed by Plis' [8]. Choose a vector ω in $\mathbb{R}^n$ of unit length, and define

$$b_\alpha(x) = \sum_0^\infty 2^{-j(m-|\alpha|)} a_\alpha(2^{-j}x - 4\omega)$$

for each α with $|\alpha| \leqslant m$. The functions $b_\alpha(x)$ are well-defined and, in fact, infinitely differentiable, since at most one term in the sum on the right side can fail to vanish at any point x in $\mathbb{R}^n$. Furthermore, since the functions $b_\alpha(x)$ differ from zero only if x is such that there exists a nonnegative integer j for which $|x - 2^{j+2}\omega| \leqslant 2^j$, it is easily verified that

$$|x|^{m-|\alpha|} |b_\alpha(x)| \leqslant 5^{m-|\alpha|}[\max_{x \in \mathbb{R}^n} |a_\alpha(x)|]$$

for each x in $\mathbb{R}^n$ and for each α with $|\alpha| \leqslant m$.

Now define

$$Au(x) = A_\infty u(x) + \sum_{|\alpha| \leqslant m} b_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

and

$$v_j(x) = u_0(2^{-j}x - 4\omega)$$

for $j = 0, 1, 2, \ldots$. One immediately sees that

$$Av_j(x) = 0$$

for $j = 0, 1, \ldots$. Since the functions v_j are infinitely differentiable and have nonoverlapping compact supports in $\mathbb{R}^n$, it follows that the null space of A in each Banach space $H_{m,p}(\mathbb{R}^n; \mathbb{C}^k)$ is infinite-dimensional.

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