JOURNAL OF DIFFERENTIAL EQUATIONS 23, 459–471 (1977)

Some Remarks on the Local Energy Decay of Solutions of the Initial-Boundary Value Problem for the Wave Equation in Unbounded Domains

HOMER F. WALKER

Department of Mathematics, University of Houston, Houston, Texas, 77004

Received September 20, 1973

1. INTRODUCTION

In this note, we consider the initial-boundary value problem for the wave equation

$$\left.\begin{aligned} u_{tt}(x,t) - \Delta u(x,t) &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \qquad \text{for} \quad x = (x_1, x_2, x_3) \in \mathscr{D} \subseteq \mathbb{R}^3 \qquad (*)$$

$$u(x,t) = 0 \quad \text{for } x \in \partial\mathscr{D}$$

with initial data $F = \{f, g\}$ in $\mathscr{H}(\mathscr{D})$, the Hilbert space of finite-energy initial data obtained by completing the space of $C_0^\infty(\mathscr{D})$-pairs $\Phi = \{\phi, \psi\}$ with respect to the energy norm

$$\| \Phi \|_{E(\mathscr{D})} = \left\{ \int_{\mathscr{D}} | \nabla\phi |^2 + | \psi |^2 \right\}^{1/2}.$$

It is assumed that $\partial\mathscr{D}$ is smooth, in which case $(*)$ admits a unique solution $u(x, t)$ which is such that the pair $U(t)F = \{u, u_t\}$ is in $\mathscr{H}(\mathscr{D})$ for each time t. In fact, it is shown in [4] that the operators $U(t)$ comprise the group of iuntary operators on $\mathscr{H}(\mathscr{D})$ generated by $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$, a skew-selfadjoint operator on $\mathscr{H}(\mathscr{D})$. It follows that $\| U(t)F \|_{E(\mathscr{D})} = \| F \|_{E(\mathscr{D})}$ for all $F \in \mathscr{H}(\mathscr{D})$ and all time t, i.e., the total energy over $\mathscr{D}$ of the solution of $(*)$ is conserved.

A number of authors have studied the behavior for large time of the *local* energy norm

$$\| U(t)F \|_{E(K)} = \left\{ \int_K | \nabla u |^2 + | u_t |^2 \right\}$$

of the solution of $(*)$ over a compact subset K of $\mathscr{D}$. Morawetz [5, 6] showed that if $\mathscr{D}$ is the exterior of a finite star-shaped body, then there exists a uniform estimate of local energy decay

$$\| U(t)F \|_{E(K)} \leqslant P(t) \| F \|_{E(\mathscr{D})}$$

ISSN 0022–0396

for all $F \in \mathscr{H}(\mathscr{D})$ with support in K, where $P(t)$ approaches zero like a power of t^{-1} as t grows large. An improved (exponential) rate of uniform local energy decay in this case was later obtained by Lax, Morawetz, and Phillips [2]. Zachmanoglou [8] extended the earlier work of Morawetz, establishing that local energy decays uniformly like a power of t^{-1} in an unbounded region with star-shaped complement. He also gave examples in [9] of slow (nonexponential) local energy decay in such domains.

The investigation of local energy decay in general exterior domains was begun by Lax and Phillips [3, 4], who proved that if $\mathscr{D}$ is the exterior of an arbitrary bounded obstacle with smooth boundary, then $\lim_{t\to\infty} \| U(t)F \|_{E(K)} = 0$ for all $F \in \mathscr{H}(\mathscr{D})$. They also conjectured that there can be no *uniform* rate of local energy decay in the exterior of "confining" obstacles, i.e., obstacles which "trap" reflected bicharacteristics of the wave equation inside a bounded region in space for arbitrarily great lengths of time. Ralston [7] verified this conjecture by constructing solutions of the wave equation whose energy remains arbitrarily close to a given reflected bicharacteristic for an arbitrarily long time.

In the following, we attempt to shed further light on the local energy decay of solutions of $(*)$. In the next section, we assume that $\mathscr{D}$ is any domain in which local energy decay occurs and we prove a theorem about the qualitative nature of this decay. In the third section, we prove a quantitative version of this result in two special domains whose simple geometry permits an explicit representation of the solution of $(*)$. In the final section, we construct an example of slow local energy decay. It is clear that most of the results of this paper are valid, either as they stand or in slightly altered form, for many other initial-boundary value problems.

2. The Dependence of Local Energy Decay on the Smoothness of the Initial Data

We now assume that $\mathscr{D}$ is a domain in $\mathbb{R}^3$ which is such that the local energy of each finite-energy solution of $(*)$ decays over a given compact subset K of $\mathscr{D}$, i.e., $\lim_{t\to\infty} \| U(t)F \|_{E(K)} = 0$ for all $F \in \mathscr{H}(\mathscr{D})$. For $\alpha > 0$, denote

$$\mathscr{H}_0^\alpha(K) = \{F = \{f, g\} \in \mathscr{H}(\mathscr{D}) : f \in H_0^{1+\alpha}(K) \quad \text{and} \quad g \in H_0^\alpha(K)\},$$

where, for $\beta > 0$, $H_0^\beta(K)$ denotes the Hilbert space of equivalence classes of functions with support in K having square-integrable fractional derivatives of all orders less than or equal to β. ($H_0^\beta(K)$ may be defined in several equally satisfactory ways.) $\mathscr{H}_0^\alpha(K)$ is a Hilbert space with norm

$$\| F \|_{K,\alpha} = \{\| f \|^2_{H_0^{1+\alpha}(K)} + \| g \|^2_{H_0^\alpha(K)}\}^{1/2}.$$

For $t \geqslant 0$, define

$$P_{K,\alpha}(t) = \sup_{\substack{F \in \mathscr{H}_0^\alpha(K) \\ F \neq 0}} \frac{\| U(t)F \|_{E(K)}}{\| F \|_{K,\alpha}}.$$

This is well defined, for if $F \in \mathscr{H}_0^\alpha(K)$, then $\| U(t)F \|_{E(K)} \leqslant \| F \|_{E(\mathscr{D})} \leqslant C \| F \|_{K,\alpha}$ for an appropriate constant C depending on α. It is clear that

$$\| U(t)F \|_{E(K)} \leqslant P_{K,\alpha}(t) \| F \|_{K,\alpha}$$

for all $t \geqslant 0$ and all $F \in \mathscr{H}_0^\alpha(K)$.

THEOREM. $\lim_{t\to\infty} P_{K,\alpha}(t) = 0$.

Proof. Suppose the contrary, i.e., suppose that there exists a positive ϵ and a sequence $\{t_n\}$, with $t_n \to \infty$, such that $P_{K,\alpha}(t_n) \geqslant \epsilon$. Then, for each n, there exists an element F_n of $\mathscr{H}_0^\alpha(K)$ with $\| F_n \|_{E(\mathscr{D})} = 1$ and with $\| U(t_n)F_n \|_{E(K)} \geqslant (\epsilon/2) \| F_n \|_{K,\alpha}$. It follows that the sequence $\{F_n\}$ is bounded in $\mathscr{H}_0^\alpha(K)$, and, by the Rellich Compactness Theorem [1], there exists a subsequence $\{F_{n_j}\}$ which converges in the norm $\| \; \|_{E(\mathscr{D})}$ to an element F^* of $\mathscr{H}(\mathscr{D})$ with support in K.

One has that

$$\begin{aligned} &\| F^* - F_{n_j} \|_{E(\mathscr{D})} + \| U(t_{n_j})F^* \|_{E(K)} \\ &\quad \geqslant \| U(t_{n_j})F^* - U(t_{n_j})F_{n_j} \|_{E(K)} + \| U(t_{n_j})F^* \|_{E(K)} \\ &\quad \geqslant \| U(t_{n_j})F_{n_j} \|_{E(K)} \geqslant \frac{\epsilon}{2} \| F_{n_j} \|_{K,\alpha}. \end{aligned}$$

Since $\lim_{j\to\infty} \| F^* - F_{n_j} \|_{E(\mathscr{D})} = 0$ and $\lim_{j\to\infty} \| U(t_{n_j})F^* \|_{E(K)} = 0$, it follows that $\lim_{j\to\infty} \| F_{n_j} \|_{K,\alpha} = 0$. But this is a contradiction, since $C \| F_{n_j} \|_{K,\alpha} \geqslant \| F_{n_j} \|_{E(\mathscr{D})} = 1$, and the theorem is proved.

In a private communication, P. D. Lax has suggested an equally easy proof of this theorem. His proof is based on the observation that, by the Rellich Compactness Theorem, the unit sphere in $\mathscr{H}_0^\alpha(K)$ is relatively compact in $\mathscr{H}(\mathscr{D})$ and, hence, is totally bounded in $\mathscr{H}(\mathscr{D})$.

3. A Quantitative Estimate of Local Energy Decay in two Special Domains

We will now give a quantitative version of the theorem of the preceding section in two domains whose simple geometry permits an explicit representation of the solution of $(*)$. The domains are $\mathscr{D}_1 = \{x \in \mathbb{R}^3 : 0 \leqslant x_3 \leqslant \pi\}$ and $\mathscr{D}_2 = \{x \in \mathbb{R}^3 : 0 \leqslant x_2 \leqslant \pi, 0 \leqslant x_3 \leqslant \pi\}$. Without loss of generality, we consider only initial data with support in $K \equiv \{x \in \mathbb{R}^3 : 0 \leqslant x_j \leqslant \pi, j = 1, 2, 3\}$, a compact subset of both $\mathscr{D}_1$ and $\mathscr{D}_2$. For convenience we take, for $\beta > 0$, $H_0{}^\beta(K)$ to be the completion of functions $\phi \in C_0{}^\infty(K)$ (expressible as a Fourier sine series

$$\phi(x) = \sum_{\substack{p=(p_1,p_2,p_3)\\ 1\leqslant p_j<\infty}} c_p \sin p_1x_1 \sin p_2x_2 \sin p_3x_3$$

in K) with respect to the norm

$$\|\phi\|_{H_0^\beta(K)} = \Bigg\{ \sum_{\substack{p=(p_1,p_2,p_3)\\ 1\leqslant p_j<\infty}} (1 + |p|^2)^\beta |c_p|^2 \Bigg\}^{1/2}.$$

THEOREM. *For $j = 1, 2$ and $0 < \alpha \leqslant 1$, there exists a calculable constant C, depending on j and α, for which $\| U(t)F\|_{E(K)} \leqslant Ct^{-\alpha/j} \|F\|_{K,\alpha}$ for all $F \in \mathscr{H}_0{}^\alpha(K)$.*

Proof. For simplicity, we consider only initial data of the form $F = \{f, 0\}$ in $\mathscr{H}_0{}^\alpha(K)$. It is convenient to work with the solution $u(x, t)$ of $(*)$, rather than with $U(t)F$.

First, suppose that $j = 1$. Setting $X = (x_1, x_2)$, we have the Fourier sine series representation

$$u(x, t) = \sum_{1\leqslant n<\infty} a_n(X, t) \sin nx_3,$$

where each function $a_n(X, t)$ solves the initial value problem

$$\begin{aligned} a_{n_{tt}} - a_{n_{x_1x_1}} - a_{n_{x_2x_2}} + n^2 a_n &= 0 \\ a_n(X, 0) = f_n(X) &\equiv \frac{2}{\pi}\int_0^\pi f(x) \sin nx_3 \, dx_3 \\ a_{n_t}(X, 0) &= 0. \end{aligned}$$

By taking Fourier transforms in X and solving the resulting ordinary differential equation in t, we obtain

$$a_n(X, t) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{iX\xi} \hat{f}_n(\xi) \cos \lambda_n(\xi)t \, d\xi,$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\lambda_n(\xi) = (|\xi|^2 + n^2)^{1/2}$, and

$$\hat{f}_n(\xi) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-X\xi} f_n(X)\, dX.$$

Setting $K_1 = [0, \pi] \times [0, \pi]$, we observe that

$$\int_K \{|\nabla u|^2 + |u_t|^2\}\, dx = \sum_{1\leqslant n<\infty} \frac{\pi}{2} \int_{K_1} \{|a_{n_{x_1}}|^2 + |a_{n_{x_2}}|^2 + n^2 |a_n|^2 + |a_{n_t}|^2\}\, dX.$$

From this, it is easily seen that the theorem follows from the following inequalities: There exist constants C for which

$$\int_{K_1} | a_{n_{x_k}} |^2 \, dX \leqslant \frac{C}{t^2} \{\| f_{n_{x_1}} \|^2 + \| f_{n_{x_2}} \|^2 + n^2 \| f_n \|^2\} \qquad \text{for} \quad k = 1, 2, \tag{1.1}$$

$$\int_{K_1} | a_n |^2 \, dX \leqslant C \frac{n^{2\alpha}}{t^{2\alpha}} \| f_n \|^2, \tag{1.2}$$

$$\int_{K_1} | a_{n_t} |^2 \, dX \leqslant C \frac{n^{2\alpha}}{t^{2\alpha}} \{\| f_{n_{x_1}} \|^2 + \| f_{n_{x_2}} \|^2 + n^2 \| f_n \|^2\}, \tag{1.3}$$

where now (and until further notice) $\| \ \|$ denotes the $L_2(\mathbb{R}^2)$-norm.

Now the inequalities (1.1)–(1.3) certainly hold with $C = 1$ whenever $n/t \geqslant 1$. Furthermore, if $n/t < 1$, then $n^2/t^2 \leqslant n^{2\alpha}/t^{2\alpha}$. Consequently, to prove the theorem in the case $j = 1$, it suffices to establish inequalities (1.1)–(1.3) with $\alpha = 1$. In the following proofs of (1.1)–(1.3) with $\alpha = 1$, we suppress the subscript n and write $a_n = a$, $f_n = f$, and $\lambda_n = \lambda$.

To prove (1.1), we observe that

$$\int_{K_1} | a_{x_k} |^2 \, dX = \sup_{\substack{\phi \in C_0^\infty(K_1) \\ \|\phi\| \leqslant 1}} \left| \int_{\mathbb{R}^2} \phi \bar{a}_{x_k} \, dX \right|^2,$$

and, for $\phi \in C_0^\infty(K_1)$ with $\| \phi \| \leqslant 1$, Parseval's relation yields

$$\int_{\mathbb{R}^2} \phi \bar{a}_{x_k} \, dX = -i \int_{\mathbb{R}^2} \hat{\phi} \xi_k \bar{\hat{f}} \cos \lambda t \, d\xi = -\frac{i}{t} \int_{\mathbb{R}^2} \hat{\phi} \lambda \bar{\hat{f}} \frac{\partial}{\partial \xi_k} \sin \lambda t \, d\xi$$

$$= \frac{i}{t} \int_{\mathbb{R}^2} \{\hat{\phi}_{\xi_k} \lambda \bar{\hat{f}} + \hat{\phi} \frac{\xi_k}{\lambda} \bar{\hat{f}} + \hat{\phi} \lambda \bar{\hat{f}}_{\xi_k}\} \sin \lambda t \, d\xi.$$

Since $\| \phi \| \leqslant 1$ and ϕ and f vanish outside K_1, applications of Schwarz's inequality and Parseval's relation give

$$\left| \int_{\mathbb{R}^2} \phi \bar{a}_{x_k} \, dX \right| \leqslant \frac{C}{t} \{\| f_{x_1} \| + \| f_{x_2} \| + n \| f \|\}$$

for an appropriate constant C, and (1.1) follows.

To prove (1.2) we again take $\phi \in C_0^\infty(K_1)$ with $\| \phi \| \leqslant 1$ and apply Parseval's relation to obtain

$$\int_{\mathbb{R}^2} \phi \bar{a} \, dX = \int_{\mathbb{R}^2} \hat{\phi} \bar{\hat{f}} \cos \lambda t \, d\xi.$$

In polar coordinates, the second integral becomes

$$\int_0^{2\pi}\int_0^\infty \hat{\phi}\bar{\hat{f}} \cos \lambda t\, \rho\, d\rho\, d\theta$$

$$= \frac{1}{t}\int_0^{2\pi}\int_0^\infty \hat{\phi}\bar{\hat{f}}\lambda \frac{\partial}{\partial\rho} \sin \lambda t\, d\rho\, d\theta$$

$$= \frac{1}{t}\int_0^{2\pi} \hat{\phi}(0)\bar{\hat{f}}(0)\,\lambda(0) \sin \lambda(0)t\, d\theta - \frac{1}{t}\int_0^{2\pi}\int_0^\infty \left\{\hat{\phi}_\rho\bar{\hat{f}}\lambda + \hat{\phi}\bar{\hat{f}}_\rho\lambda + \hat{\phi}\bar{\hat{f}}\frac{\rho}{\lambda}\right\} \sin \lambda t\, d\rho\, d\theta$$

$$= \frac{2\pi n}{t}\hat{\phi}(0)\bar{\hat{f}}(0) \sin nt - \frac{1}{t}\int_0^{2\pi}\int_0^1 \left\{\hat{\phi}_\rho\bar{\hat{f}}\lambda + \hat{\phi}\bar{\hat{f}}_\rho\lambda + \hat{\phi}\bar{\hat{f}}\frac{\rho}{\lambda}\right\} \sin \lambda t\, d\rho\, d\theta$$

$$- \frac{1}{t}\int_{|\xi|\geqslant 1} \left\{\nabla\hat{\phi}\cdot\frac{\xi}{|\xi|}\bar{\hat{f}}\lambda + \hat{\phi}\nabla\bar{\hat{f}}\cdot\frac{\xi}{|\xi|}\lambda + \hat{\phi}\bar{\hat{f}}\frac{|\xi|}{\lambda}\right\}\frac{\sin \lambda t}{|\xi|}\, d\xi.$$

Again drawing on the fact that $\|\phi\| \leqslant 1$ and ϕ and f vanish outside K_1, we obtain an estimate

$$\left|\int_{\mathbb{R}^2} \phi\bar{a}\, dX\right| \leqslant C\frac{n}{t}\|f\|$$

for an appropriate constant C, and (1.2) follows.

To prove (1.3), we observe that

$$\int_{\mathbb{R}^2} \phi\bar{a}_t\, dx = -\int_{\mathbb{R}^2} \hat{\phi}\bar{\hat{f}}\lambda \sin \lambda t\, d\xi$$

for $\phi \in C_0^\infty(K_1)$ with $\|\phi\| \leqslant 1$. Proceeding as in the proof of (1.2), with $\lambda\bar{\hat{f}}$ replacing $\bar{\hat{f}}$, one derives (1.3). This completes the proof of the theorem in the case $j = 1$.

Now suppose that $j = 2$. The Fourier sine series representation of the solution of $(*)$ is now

$$u(x, t) = \sum_{\substack{1\leqslant n<\infty\\ 1\leqslant m<\infty}} a_{nm}(x_1, t) \sin nx_2 \sin mx_3$$

with

$$a_{nm}(x_1, t) = (2\pi)^{-1/2}\int_{\mathbb{R}^1} e^{ix_1\xi}\hat{f}_{nm}(\xi) \cos \lambda_{nm}(\xi)t\, d\xi,$$

where

$$f_{nm}(x_1) = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x) \sin nx_2 \sin mx_3\, dx_2\, dx$$

$$\hat{f}_{nm}(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}^1} e^{-ix_1\xi} f_{nm}(x_1)\, dx_1,$$

and

$$\lambda_{nm}(\xi) = (\xi^2 + n^2 + m^2)^{1/2} \qquad \text{for} \quad \xi \in \mathbb{R}^1.$$

Observing that

$$\int_K \{|\nabla u|^2 + |u_t|^2\}\,dx = \sum_{\substack{1\leq n<\infty\\ 1\leq m<\infty}} \frac{\pi^2}{4}\int_0^\pi \{|a_{nm_{x_1}}|^2 + (n^2+m^2)\,|a_{nm}|^2 + |a_{nm_t}|^2\}\,dx_1,$$

one sees that the theorem follows in the case $j = 2$ from the following inequalities: There exist constants C for which

$$\int_0^\pi |a_{nm_{x_1}}|^2\,dx_1 \leq \frac{C}{t^2}\{\|f_{nm_{x_1}}\|^2 + (n^2+m^2)\|f_{nm}\|^2\}, \tag{2.1}$$

$$\int_0^\pi |a_{nm}|^2\,dx_1 \leq C\,\frac{(n^2+m^2)^\alpha}{t^\alpha}\,\|f_{nm}\|^2, \tag{2.2}$$

$$\int_0^\pi |a_{nm_t}|^2\,dx_1 \leq C\,\frac{(n^2+m^2)^\alpha}{t^\alpha}\{\|f_{nm_{x_1}}\|^2 + (n^2+m^2)\,\|f_{nm}\|^2\}, \tag{2.3}$$

where now (and for the remainder of the proof) $\|\ \|$ denotes the $L_2(\mathbb{R}^1)$ norm. As before, it suffices to prove (2.1)–(2.3) only in the case $\alpha = 1$. Again suppressing subscripts, we write $a_{nm} = a$, $f_{nm} = f$, and $\lambda_{nm} = \lambda$. For convenience, we assume $t \geq 1$.

The proof of (2.1) proceeds like that of (1.1), and so we omit it. To prove (2.2), we observe that

$$\int_0^\pi |a_{nm}|^2\,dx_1 = \sup_{\substack{\phi\in C_0^\infty([0,\pi])\\ \|\phi\|\leq 1}} \left|\int_{\mathbb{R}^1} \phi\bar{a}\,dx_1\right|^2,$$

and, from Parseval's relation

$$\int_{\mathbb{R}^1} \phi\bar{a}\,dx_1 = \int_{\mathbb{R}^1} \hat{\phi}\bar{\hat{f}}\cos\lambda t\,d\xi = \int_{-1}^1 \hat{\phi}\bar{\hat{f}}\cos\lambda t\,d\xi + \int_{|\xi|\geq 1} \hat{\phi}\bar{\hat{f}}\cos\lambda t\,d\xi$$

for $\phi \in C_0^\infty[0, \pi]$ with $\|\phi\| \leq 1$. Set $\mu = (m^2 + n^2)^{1/2}$.

Claim 1. $|\int_{-1}^1 \hat{\phi}\bar{\hat{f}}\cos\lambda t\,d\xi| \leq C(\mu/t^{1/2})\,\|f\|$ for an appropriate constant C.

Proof of Claim 1. We will show that

$$\left|\int_0^1 \hat{\phi}\bar{\hat{f}}\cos\lambda t\,d\xi\right| \leq C\,\frac{\mu}{t^{1/2}}\,\|f\|.$$

It follows similarly that

$$\left| \int_{-1}^{0} \hat{\phi} \bar{f} \cos \lambda t \, d\xi \right| \leqslant C \frac{\mu}{t^{1/2}} \|f\|.$$

For a given t, let $k = k(t)$ be the integer such that $((k-2)\pi)/t < \mu \leqslant ((k-1)\pi)/t$. Set $\alpha_0 = ((k\pi/t)^2 - \mu^2)^{1/2}$. Note that

$$\alpha_0 \leqslant \left(\left(\mu + \frac{2\mu}{t}\right)^2 - \mu^2\right)^{1/2} \leqslant \left(\frac{4\pi\mu}{t} + \left(\frac{2\pi}{t}\right)^2\right)^{1/2} \leqslant C\left(\frac{\mu}{t}\right)^{1/2}$$

for an appropriate constant C.

Case 1. Suppose that t is so small that $\alpha_0 \geqslant \frac{1}{2}$. It follows that

$$\left| \int_{0}^{1} \hat{\phi} \bar{f} \cos \lambda t \, d\xi \right| \leqslant 2\alpha_0 \sup |\hat{\phi} \bar{f}| \leqslant C \left(\frac{\mu}{t}\right)^{1/2} \|f\| \leqslant C \frac{\mu}{t^{1/2}} \|f\|,$$

and Claim 1 is proved in this case.

Case 2. If t is so large that $\alpha_0 \leqslant \frac{1}{2}$, define

$$\eta(\xi) = \left(\xi^2 + \frac{2\pi\lambda(\xi)}{t} + \left(\frac{\pi}{t}\right)^2\right)^{1/2}$$

for

$$\xi \geqslant \alpha_1 \equiv \left(\left(\frac{(k-1)\pi}{t}\right)^2 - \mu^2\right)^{1/2}.$$

Note that $\eta(\alpha_1) = \alpha_0$, and, for all $\xi \geqslant \alpha_1$, $\xi < \eta(\xi)$ and $\lambda(\eta(\xi)) = \lambda(\xi) + \pi/t$. Since $\alpha_0 \leqslant \frac{1}{2}$, we have

$$\frac{1}{4} \geqslant \alpha_0^2 = \left(\frac{k\pi}{t}\right)^2 - \mu^2 \geqslant \left(\mu + \frac{\pi}{t}\right)^2 - \mu^2 = \frac{2\mu\pi}{t} + \left(\frac{\pi}{t}\right)^2 \geqslant \frac{2\pi(1+\mu^2)^{1/2}}{2^{1/2}t} - \frac{1}{2^{1/2}}\left(\frac{\pi}{t}\right)^2.$$

Consequently, $1 - (2\pi/t)(1+\mu^2)^{1/2} + (\pi/t)^2 \geqslant 1 - (2^{1/2}/4) > 0$, and

$$\alpha_2 \equiv \left(1 - \frac{2\pi}{t}(1+\mu^2)^{1/2} + \left(\frac{\pi}{t}\right)^2\right)^{1/2}$$

is a well-defined real number. In fact,

$$\alpha_2 \geqslant \left(1 - \frac{2^{1/2}}{4}\right) > \frac{1}{2} \geqslant \alpha_0 > \alpha_1,$$

and $\eta(\alpha_2) = 1$.

Now one has

$$\int_0^1 \hat{\phi}\bar{f} \cos \lambda t \, d\xi = \int_0^{\alpha_0} \hat{\phi}\bar{f} \cos \lambda t \, d\xi + \frac{1}{2} \int_{\alpha_0}^1 \hat{\phi}(\xi) \bar{f}(\xi) \cos \lambda(\xi) t \, d\xi$$

$$+ \frac{1}{2} \int_{\alpha_1}^{\alpha_2} \hat{\phi}(\eta(\xi)) \bar{f}(\eta(\xi)) \cos \lambda(\eta(\xi)) t \, \frac{d\eta}{d\xi}(\xi) \, d\xi$$

$$= \int_0^{\alpha_0} \hat{\phi}\bar{f} \cos \lambda t \, d\xi + \frac{1}{2} \int_{\alpha_2}^1 \hat{\phi}\bar{f} \cos \lambda t \, d\xi$$

$$- \frac{1}{2} \int_{\alpha_1}^{\alpha_0} \hat{\phi}(\eta(\xi)) \bar{f}(\eta(\xi)) \cos \lambda(\xi) t \, \frac{d\eta}{d\xi} \, d\xi$$

$$+ \frac{1}{2} \int_{\alpha_0}^{\alpha_2} \left\{ \hat{\phi}(\xi) \bar{f}(\xi) - \hat{\phi}(\eta(\xi)) \bar{f}(\eta(\xi)) \frac{d\eta}{d\xi} \right\} \cos \lambda(\xi) t \, d\xi.$$

Since $\alpha_0 \leqslant C(\mu/t)^{1/2}$ and $\alpha_0 - \alpha_1 \leqslant \alpha_0$ and $d\eta/d\xi = \xi(\lambda + \pi/t)/\lambda\eta(\xi)$, one has that

$$\left| \int_0^{\alpha_0} \hat{\phi}\bar{f} \cos \lambda t \, d\xi \right| \leqslant C \left(\frac{\mu}{t} \right)^{1/2} \sup |\hat{\phi}\bar{f}| \leqslant C \left(\frac{\mu}{t} \right)^{1/2} \|f\| \tag{3}$$

and

$$\left| \frac{1}{2} \int_{\alpha_1}^{\alpha_0} \hat{\phi}(\eta(\xi)) \bar{f}(\eta(\xi)) \cos \lambda(\xi) t \, \frac{d\eta}{d\xi} \, d\xi \right| \leqslant C \left(\frac{\mu}{t} \right)^{1/2} \sup |\hat{\phi}\bar{f}| \leqslant C \left(\frac{\mu}{t} \right)^{1/2} \|f\|. \tag{4}$$

Also, since $1 \geqslant \alpha_2 > \frac{1}{2}$, one has that

$$1 - \alpha_2 = 1 - \left(1 - \frac{2\pi}{t} (1 + \mu^2)^{1/2} + \left(\frac{\pi}{t} \right)^2 \right)^{1/2}$$

$$\leqslant \left\{ \frac{2\pi}{t} (1 + \mu^2)^{1/2} - \left(\frac{\pi}{t} \right)^2 \right\} \frac{1}{2\alpha_2} \leqslant C \frac{\mu}{t},$$

and it follows that

$$\left| \frac{1}{2} \int_{\alpha_2}^1 \hat{\phi}\bar{f} \cos \lambda t \, d\xi \right| \leqslant C \frac{\mu}{t} \sup |\hat{\phi}\bar{f}| \leqslant C \frac{\mu}{t} \|f\|. \tag{5}$$

Now setting $\zeta(\epsilon) = (\xi^2 + 2\lambda(\xi)\epsilon + \epsilon^2)^{1/2}$, one has that

$$\hat{\phi}(\xi) \bar{f}(\xi) - \hat{\phi}(\eta(\xi)) \bar{f}(\eta(\xi)) \frac{d\eta}{d\xi}$$

$$= - \int_0^{\pi/t} \frac{d}{d\epsilon} \left\{ \hat{\phi}(\zeta(\epsilon)) \bar{f}(\zeta(\epsilon)) \frac{\xi(\lambda + \epsilon)}{\lambda\zeta(\epsilon)} \right\} d\epsilon$$

$$= - \int_0^{\pi/t} \{ [\hat{\phi}_\xi(\zeta(\epsilon)) \bar{f}(\zeta(\epsilon)) + \hat{\phi}(\zeta(\epsilon)) \bar{f}_\xi(\zeta(\epsilon))] \frac{\xi(\lambda + \epsilon)^2}{\lambda\zeta(\epsilon)^2}$$

$$+ \hat{\phi}(\zeta(\epsilon)) \bar{f}(\zeta(\epsilon)) \left[\frac{-\xi\mu^2}{\lambda\zeta(\epsilon)^3} \right] \} \, d\epsilon.$$

So

$$\left| \hat{\phi}(\xi) \bar{f}(\xi) - \hat{\phi}(\eta(\xi)) \bar{f}(\eta(\xi)) \frac{d\eta}{d\xi} \right|$$

$$\leqslant [\max\{\sup |\hat{\phi}|, \sup |\hat{\phi}_\xi|\}] [\max\{\sup |\bar{f}|,$$

$$\sup |\hat{f}_\xi|\}] \cdot \frac{C}{t} \left\{\frac{\lambda}{\xi} + \frac{\mu}{\xi^2}\right\}$$

$$\leqslant \frac{C}{t} \left\{1 + \frac{\mu}{\xi} + \frac{\mu}{\xi^2}\right\} \|f\|.$$

Consequently,

$$\left| \frac{1}{2} \int_{\alpha_0}^{\alpha_2} \left\{\hat{\phi}(\xi) \bar{f}(\xi) - \hat{\phi}(\eta(\xi)) \bar{f}(\eta(\xi)) \frac{d\eta}{d\xi}\right\} \cos \lambda(\xi)\, t d\xi \right|$$

$$\leqslant \frac{C}{t} \|f\| \left\{\alpha_2 - \alpha_0 + \mu(\log \alpha_2 - \log \alpha_0) + \mu \left(\frac{1}{\alpha_0} - \frac{1}{\alpha_2}\right)\right\}$$

$$\leqslant C \frac{\mu}{t} \{1 + \log t + t^{1/2}\} \|f\|. \tag{6}$$

It follows from (3)–(6) that

$$\left| \int_0^1 \hat{\phi} \bar{f} \cos \lambda t \, d\xi \right| \leqslant C(\mu/t^{1/2}) \|f\|,$$

and the proof of Claim 1 is complete.

Claim 2.

$$\left| \int_{|\xi| \geqslant 1} \hat{\phi} \bar{f} \cos \lambda t \, d\xi \right| \leqslant C \frac{\mu}{t} \|f\|.$$

Proof of Claim 2. One has that

$$\int_{|\xi|\geqslant 1} \hat{\phi}\bar{f} \cos \lambda t \, d\xi = \frac{1}{t} \int_{|\xi|\geqslant 1} \frac{\lambda}{\xi} \hat{\phi}\bar{f} \frac{d}{d\xi} \sin \lambda t \, d\xi$$

$$= -\frac{1}{t} \lambda(1) \sin \lambda(1)\, t\{\hat{\phi}(-1) \bar{f}(-1) + \hat{\phi}(1) \bar{f}(1)\}$$

$$-\frac{1}{t} \int_{|\xi|\geqslant 1} \left\{\left(\frac{1}{\lambda} - \frac{\lambda}{\xi^2}\right) \hat{\phi}\bar{f} + \frac{\lambda}{\xi} (\hat{\phi}_\xi \bar{f} + \hat{\phi}\bar{f}_\xi)\right\} \sin \lambda t \, d\xi.$$

Consequently,

$$\left| \int_{|\xi|\geqslant 1} \hat{\phi}\bar{f} \cos \lambda t \, d\xi \right| \leqslant C \frac{\mu}{t} \left\{\sup |\hat{\phi}\bar{f}| + \int_{\mathbb{R}^1} [|\hat{\phi}\bar{f}| + |\hat{\phi}_\xi \bar{f}| + |\hat{\phi}\bar{f}_\xi|] \, d\xi\right\}$$

$$\leqslant C \frac{\mu}{t} \|f\|,$$

and Claim 2 is proved.

It follows from Claims 1 and 2 that

$$\int_0^\pi |a|^2\,dx_1 \leqslant C\,\frac{(n^2+m^2)}{t}\,\|f\|^2,$$

and the proof of (2.2) is complete.

To prove (2.3), we observe that

$$\int_{\mathbb{R}^1} \phi\bar{a}_t\,dx_1 = -\int_{\mathbb{R}^1} \hat{\phi}\lambda\bar{\hat{f}}\sin\lambda t\,d\xi$$

for $\phi \in C_0^\infty([0,\pi])$ with $\phi \leqslant 1$. Proceeding as in the proof of (2.2), with $\lambda\bar{\hat{f}}$ replacing $\hat{f}$, one derives (2.3). This completes the proof of the theorem.

4. An Example of Slow Local Energy Decay

In this section, we construct a solution of (*) in the domain $\mathscr{D}_1$ defined in the preceding section whose energy decays locally more slowly than any negative power of t. The initial data for this solution have support in the set K, which appears in the preceding section, and lie in $\mathscr{H}(\mathscr{D}_1)$. It is evident that the data can belong to no space $\mathscr{H}_0^\alpha(K)$ for $\alpha > 0$.

Let $\phi \in C_0^\infty(\mathscr{D}_1)$ have support in K and be such that $\phi(x) \geqslant 0$ for all $x \in \mathscr{D}_1$, $\phi(x) = 1$ for $\pi/4 \leqslant x_j \leqslant 3\pi/4$, $j = 1, 2, 3$, and $\phi_{x_3}(x) = 0$ for all $x \in \mathscr{D}_1$ with $\pi/4 \leqslant x_3 \leqslant 3\pi/4$. For $x \in \mathscr{D}_1$, define

$$g(x) = \begin{cases} \dfrac{\phi(x)\operatorname{sign}(x_3-\pi/2)}{|x_3-\pi/2|^{1/2}\,[\log|x_3-\pi/2|-2]} & \text{for } x_3 \neq \dfrac{\pi}{2} \\ 0 & \text{for } x_3 = \dfrac{\pi}{2}. \end{cases}$$

It is easily seen that $g \in L_2(\mathscr{D}_1)$, and hence, that $\{0, g\}$ is in $\mathscr{H}(\mathscr{D}_1)$.

We claim that the energy of the solution $u(x, t)$ of (*) in $\mathscr{D}_1$ with initial data $\{0, g\}$ decays no faster in K than a constant times $(\log t)^{-2}$. Specifically, it is shown below that there exists a constant $C > 0$ such that

$$\int_K |\nabla u|^2 + |u_t|^2 \geqslant C\,\frac{1}{(\log t)^2}$$

whenever $t = 2k\pi$ for a positive integer k.

We obtain a sperical-mean representation of $u(x, t)$ as follows: First, extend $g(x)$ as an odd function of x_3 to $\mathbb{R}^2 \times [-\pi, \pi]$; then, extend $g(x)$ periodically (period 2π) in x_3 to all of $\mathbb{R}^3$. Then $u(x, t)$ is given by

$$u(x,t) = \frac{1}{4\pi t}\int_{|y|=t} g(x+y)\,dS,$$

where dS is the element of surface area of a sphere of radius t in $\mathbb{R}^3$.

Suppose that $t = 2k\pi$ for a positive integer k, and suppose that

$$x \in K(t) = \left\{ x \in K\colon (x_1, x_2) \in \left[\frac{3\pi}{8}, \frac{5\pi}{8}\right] \times \left[\frac{3\pi}{8}, \frac{5\pi}{8}\right] \quad \text{and} \quad 0 < x_3 - \frac{\pi}{2} \leqslant \frac{1}{t} \right\}.$$

We have

$$u_{x_3}(x, t) = \frac{1}{4\pi t} \int_{|y|=t} g_{x_3}(x + y)\, dS,$$

and, if k is sufficiently large, it follows that

$$\begin{aligned} u_{x_3}(x, t) &\geqslant \frac{1}{4\pi t} \int_{A(x,t)} \phi(y) \left\{ \frac{-\log | y_3 - (\pi/2)|}{2 \,| y_3 - (\pi/2)|^{3/2} [\log | y_3 - (\pi/2)| - 2]^2} \right\} dS \\ &\geqslant \frac{1}{4\pi t} \int_{B(x,t)} \left\{ \frac{-\log | y_3 - (\pi/2)|}{2 \,| y_3 - (\pi/2)|^{3/2} [\log | y_3 - (\pi/2)| - 2]^2} \right\} dS \\ &\geqslant \frac{1}{4\pi t} \left(\frac{\pi}{4}\right)^2 \inf_{y \in B(x,t)} \left\{ \frac{-\log | y_3 - (\pi/2)|}{2 \,| y_3 - (\pi/2)|^{3/2} [\log | y_3 - (\pi/2)| - 2]^2} \right\}, \end{aligned}$$

where

$$\begin{aligned} A(x, t) = \{ y \in \mathscr{D}_1 : (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3 - t)^2 = t^2, \\ (y_1, y_2) \in [0, \pi] \times [0, \pi]\}, \end{aligned}$$

and

$$B(x, t) = \left\{ y \in A(x, t) : (y_1 - x_1, y_2 - x_2) \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right] \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right] \right\}.$$

From this, it is apparent that there exists a constant C, independent of k and $x \in K(t)$, such that $u_{x_3}(x, t) \geqslant C(t^{1/2}/\log t)$ for all large k and all $x \in K(t)$. Then

$$\int_K | \nabla u |^2 + | u_t |^2 \geqslant \int_{K(t)} | u_{x_3} |^2 \geqslant C^2 \left(\frac{\pi}{4}\right)^2 \frac{1}{(\log t)^2}$$

as desired.

References

1. L. Bers, F. John, and M. Schechter, "Partial Differential Equations," Proc. Summer Seminar, Boulder, Colorado, 1957, Lectures in Appl. Math., Vol. 3, Interscience, New York, 1964.
2. P. D. Lax, C. S. Morawetz, and R. S. Phillips, Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle, *Comm. Pure Appl. Math.* **16** (1963), 477–486.
3. P. D. Lax and R. S. Phillips, The wave equation in exterior domains, *Bull. Amer. Math. Soc.* **68** (1962), 47–49.

4. P. D. LAX AND R. S. PHILLIPS, "Scattering Theory," Pure and Appl. Math., Vol. 26, Academic Press, New York (1967).
5. C. S. MORAWETZ, The decay of solutions of the exterior initial-boundary value problem for the wave equation, *Comm. Pure Appl. Math.* **15** (1961), 561–568.
6. C. S. MORAWETZ, The limiting amplitude principle, *Comm. Pure Appl. Math.* **15** (1962), 349–361.
7. J. V. RALSTON, Solutions of the wave equation with localized energy, *Comm. Pure Appl. Math.* **22** (1969), 807–823.
8. E. C. ZACHMANOGLOU, The decay of solutions of the initial-boundary value problem for the wave equation in unbounded regions, *Arch. Rat. Mech. Anal.* **14** (1963), 312–325.
9. E. C. ZACHMANOGLOU, An example of slow decay of the solution of the initial-boundary value problem for the wave equation in unbounded regions, *Bull. Amer. Math. Soc.* **70** (1964), 633–636.