SIAM J. OPTIMIZATION
Vol. 4, No. 2, pp. 393–422, May 1994

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GLOBALLY CONVERGENT INEXACT NEWTON METHODS*

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Abstract. Inexact Newton methods for finding a zero of $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ are variations of Newton's method in which each step only approximately satisfies the linear Newton equation but still reduces the norm of the local linear model of F. Here, inexact Newton methods are formulated that incorporate features designed to improve convergence from arbitrary starting points. For each method, a basic global convergence result is established to the effect that, under reasonable assumptions, if a sequence of iterates has a limit point at which F' is invertible, then that limit point is a solution and the sequence converges to it. When appropriate, it is shown that initial inexact Newton steps are taken near the solution, and so the convergence can ultimately be made as fast as desired, up to the rate of Newton's method, by forcing the initial linear residuals to be appropriately small. The primary goal is to introduce and analyze new inexact Newton methods, but consideration is also given to "globalizations" of (exact) Newton's method that can naturally be viewed as inexact Newton methods.

Key words. inexact Newton methods, Newton iterative methods, truncated Newton methods, Newton's method, globally convergent Newton-like methods, global convergence analyses for Newton-like methods

AMS subject classification. 65H10

1. Introduction.

Consider the system of nonlinear equations

$$F(x) = 0, \tag{1.1}$$

where $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is continuously differentiable. To simplify the following discussion, we assume *only for the remainder of this introduction* that (1.1) has a solution x_* such that $F'(x_*)$ is invertible.

A classical algorithm for solving (1.1) is Newton's method.

ALGORITHM N (Newton's method).

LET x_0 BE GIVEN.
FOR $k = 0$ STEP 1 UNTIL "CONVERGENCE" DO:
  SOLVE $F'(x_k)\, s_k = -F(x_k)$.
  SET $x_{k+1} = x_k + s_k$.

The major strength of Newton's method lies in its local convergence properties: If x_0 is sufficiently close to x_*, then $\{x_k\}$ converges q-superlinearly to x_*. Usually, F' is Lipschitz continuous at x_*, and the convergence is q-quadratic. (See, e.g., Ortega and Rheinboldt [13, §10.2.2].)

At each stage of Newton's method, the Newton equation

$$F'(x_k)\, s_k = -F(x_k) \tag{1.2}$$

* Received by the editors March, 1991; accepted for publication (in revised form) March 29, 1993.

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must be solved. Computing the exact solution can be expensive if n is large and, for any n, may not be justified when x_k is far from a solution. Thus, one might prefer to compute some approximate solution, leading to the following algorithm.

ALGORITHM IN (inexact Newton method [4]).

LET x_0 BE GIVEN.
FOR $k = 0$ STEP 1 UNTIL "CONVERGENCE" DO:
FIND **some** $\eta_k \in [0, 1)$ AND s_k THAT SATISFY

$$\|F(x_k) + F'(x_k)\, s_k\| \leq \eta_k \|F(x_k)\|. \tag{1.3}$$

SET $x_{k+1} = x_k + s_k$.

Since $F(x_k) + F'(x_k)\, s_k$ is the residual of (1.2), as well as the local linear model of F evaluated at s_k, each η_k reflects how accurately s_k solves (1.2); however, the method of determining s_k is not otherwise restricted.[1] We say that s_k satisfying (1.3) is an *inexact Newton step* (*at the level* η_k) and refer to (1.3) as an *inexact Newton condition*.

If x_0 is sufficiently close to x_* and the η_k's are uniformly bounded below one, then a sequence of inexact Newton iterates $\{x_k\}$ converges q-linearly to x_* in a well-chosen norm. If $\lim_{k\to\infty} \eta_k = 0$, then the convergence is q-superlinear. If F' is Lipschitz continuous at x_* and $\eta_k = O\|F(x_k)\|$, then the convergence is q-quadratic. In any case, the convergence can be made as fast as that of a sequence of exact Newton iterates by taking the η_k's to be appropriately small. (See Dembo, Eisenstat, and Steihaug [4].)

The convergence of Algorithms N and IN is only *local*, i.e., the iterates need not converge if x_0 is not near a solution. There are a number of "globalizations" of Algorithm N that help to improve the likelihood of convergence from arbitrary starting points. Most modern globalizations fit within the following framework at each iteration.

- Determine an initial trial step in some way related to the Newton step.
- Test the trial step to determine whether it gives adequate progress toward a solution; if it does not, then obtain a "more conservative" trial step according to the globalization strategy and repeat the test.

Trial steps are determined in a variety of ways but typically reduce the norm of the local linear model of F; thus these globalizations provide instances of Algorithm IN. In requiring adequate progress, a decrease in $\|F\|$ is imposed at each iteration that makes convergence to a solution likely. Nevertheless, the nature of the problem may still preclude convergence, and even if the iterates converge, the limit may be a local minimizer of $\|F\|$ that is not a solution, at which F' is necessarily singular.

The purpose of this paper is to introduce and analyze *globally convergent inexact Newton methods*. These are instances of Algorithm IN that fit within the above globalization framework for Algorithm N. All use a test for adequate progress that is based directly on the norms of F and its local linear model and is particularly compatible with the inexact Newton framework. This test can be viewed as an extension to the inexact Newton context of the type of tests considered by Moré and Sorensen [12] and Shultz, Schnabel, and Byrd [16] for unconstrained optimization and by Powell [14], El Hallabi [7], and El Hallabi and Tapia [8] for general nonlinear equations. It

[1] *Newton iterative* or *truncated Newton* methods, which use iterative methods to solve (1.2) approximately, constitute an important class of inexact Newton methods. Although these methods have provided considerable motivation, the developments in this paper are valid in the more general inexact Newton setting.

requires a decrease in $\|F\|$ at each iteration that makes convergence to a solution likely, although, as above, this may be precluded by the nature of the problem. For each algorithm, we establish, among other results, a basic global convergence result to the effect that, under reasonable assumptions, if a sequence of iterates has a limit point at which F' is invertible, then that limit point is a solution of (1.1) and the sequence converges to it. These results are counterparts for general nonlinear equations of results of Moré and Sorensen [12] and Shultz, Schnabel, and Byrd [16] for unconstrained optimization. When appropriate, we show in addition that initial inexact Newton trial steps are taken near a solution; therefore, as in the local theory of Dembo, Eisenstat, and Steihaug [4], the convergence can ultimately be made as fast as desired, up to the rate of Newton's method, by choosing the initial inexact Newton levels to be appropriately small.

Our primary goal is to provide a theoretical foundation for new inexact Newton algorithms. However, the analysis here also provides a general framework for treating globalizations of Algorithm N and other Newton-like algorithms, and we explore this in a number of applications. In §2, we introduce a general *global inexact Newton method*, on which all subsequent algorithms are based. In §3, we prove global convergence results for it. These developments primarily provide a foundation for §§5–8, but in §4 we give two applications to globalized Newton-like methods that show them to be of interest in their own right. In §5, we introduce and analyze two general algorithms; these serve as paradigms for more specific algorithms in §§6–8 that can be implemented as practical algorithms. The algorithms in §6 are *backtracking methods*: In these, if an initial inexact Newton step is unsatisfactory, then shorter steps in the same direction are taken until a satisfactory step is found. The algorithms in §7 are *equality curve methods*: In these, all trial steps satisfy specified inexact Newton conditions with equality; if an initial inexact Newton step is unsatisfactory, then steps at higher inexact Newton levels (which may vary in direction as well as length) are tried until a satisfactory step is found. In §8, we give two applications of the results in §7. In §9, we conclude with a summary discussion; the reader is encouraged to consult this at any time for a more detailed overview that indicates all algorithms in the paper, summarizes their properties, and shows their interrelationships.

For work that is similar in spirit to some of the work here, although differing in major details, see Steihaug [17, Chap. 5] and Brown and Saad [3, §3]. Less closely related, more specialized work can be found in Bank and Rose [2] and Deuflhard [6].

Preliminaries. *For the remainder of the paper, we assume only that $F: \mathbf{R}^n \to \mathbf{R}^n$ is continuously differentiable.* We make no assumptions about the invertibility of F' or about the existence of solutions of (1.1) unless otherwise noted.

The norm $\|\cdot\|$ is arbitrary, except where explicitly assumed to be an inner product norm or the Euclidean norm $\|\cdot\|_2$ in §§7 and 8. We denote $N_\delta(x) \equiv \{y \mid \|y-x\| < \delta\}$ for $\delta > 0$. We let $\arg\min_{x \in S} f(x)$ denote the set of arguments that minimize $f: \mathbf{R}^n \to \mathbf{R}^1$ over $S \subseteq \mathbf{R}^n$; if this set is a singleton, then we may treat it as a point.

We use the following technical results.

LEMMA 1.1 ([13, §2.3.3]). *Assume that $F'(x)$ is invertible. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that $F'(y)$ is invertible and*

$$\|F'(y)^{-1} - F'(x)^{-1}\| < \epsilon$$

whenever $y \in N_\delta(x)$.

LEMMA 1.2 ([13, §3.2.10]). *For any x and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\|F(z) - F(y) - F'(y)(z-y)\| \leq \epsilon \|z-y\|$$

whenever $y, z \in N_\delta(x)$.

In saying that x_* is a *limit point* of a sequence $\{x_k\}$, we mean that for every $\delta > 0$, there are infinitely many values of k for which $x_k \in N_\delta(x_*)$. Note that x_* is a limit point of $\{x_k\}$ if $x_k = x_*$ for infinitely many k.

In general, the existence of an inexact Newton step s_k satisfying (1.3) depends on $F(x_k)$, $F'(x_k)$, η_k, and the norm $\|\cdot\|$. There exists an inexact Newton step for *every* $\eta_k \in [0,1)$ if and only if $F(x) \in \operatorname{range} F'(x_k)$, which always holds if $F'(x_k)$ is invertible. Recall that x is a *stationary point* of $\|F\|$ if $\|F(x)\| \le \|F(x) + F'(x)\, s\|$ for every s. There is no inexact Newton step for *any* $\eta_k \in [0,1)$ if and only if $F(x_k) \neq 0$ and x_k is a stationary point of $\|F\|$; equivalently, there exists an inexact Newton step for *some* $\eta_k \in [0,1)$ if and only if either $F(x_k) = 0$ or x_k is not a stationary point of $\|F\|$. When $\|\cdot\|$ is an inner-product norm, there exists an inexact Newton step for *some* $\eta_k \in [0,1)$ if and only if either $F(x_k) = 0$ or $F(x_k) \not\perp \operatorname{range} F'(x_k)$.

Many of the algorithms below are like Algorithm IN in calling for steps that satisfy inexact Newton conditions and perhaps other conditions but are not uniquely specified. We always assume that such suitable steps are found if they exist.

In saying that an algorithm *breaks down* at some x_k, we mean that it is somehow precluded from determining a suitable x_{k+1}. For example, Algorithm IN breaks down at some x_k if and only if there is no inexact Newton step from x_k, equivalently, if and only if $F(x_k) \neq 0$ and x_k is a stationary point of $\|F\|$. To discuss convergence in the following, we state algorithms so that they continue indefinitely if they do not break down. Thus, in saying that an algorithm *does not break down*, we mean simply that it generates an infinite sequence of iterates. In most cases below, we give conditions under which specific algorithms do not break down.

2. A global inexact Newton method. Our general global inexact Newton method below is obtained by augmenting the inexact Newton condition with a sufficient decrease condition on $\|F\|$. This provides the test for acceptability of an inexact Newton step that is used, in one form or another, in all of our algorithms.

ALGORITHM GIN (global inexact Newton method).

LET x_0 AND $t \in (0,1)$ BE GIVEN.
FOR $k = 0$ STEP 1 UNTIL ∞ DO:
FIND **some** $\eta_k \in [0,1)$ AND s_k THAT SATISFY

$$\|F(x_k) + F'(x_k)\, s_k\| \le \eta_k \|F(x_k)\| \tag{2.1}$$

AND

$$\|F(x_k + s_k)\| \le [1 - t(1 - \eta_k)]\, \|F(x_k)\|. \tag{2.2}$$

SET $x_{k+1} = x_k + s_k$.

Conditions (2.1) and (2.2) are closely related to certain tests for accepting a step considered elsewhere. For minimizing $f : \mathbf{R}^n \to \mathbf{R}^1$, the central feature of those tests is that, for a given $t \in (0,1)$, a step s_k from a current point x_k is acceptable if

$$ared_k(s_k) \ge t \cdot pred_k(s_k), \tag{2.3}$$

where $ared_k(s_k) \equiv f(x_k) - f(x_k + s_k)$ is the *actual reduction* and $pred_k(s_k)$ is the *predicted reduction* obtained from a local quadratic model of f (see Moré and Sorensen [12], Shultz, Schnabel, and Byrd [16], and the references therein). For general nonlinear equations, a special case of more general tests considered by Powell [14], El

Hallabi [7], and El Hallabi and Tapia [8] has the form (2.3), with

$$\begin{aligned} ared_k(s_k) &\equiv \|F(x_k)\| - \|F(x_k+s_k)\|, \\ pred_k(s_k) &\equiv \|F(x_k)\| - \|F(x_k)+F'(x_k)s_k\|, \end{aligned} \tag{2.4}$$

which we take as our definition of $ared_k(s_k)$ and $pred_k(s_k)$ here. With (2.4), conditions (2.1) and (2.2) can be rephrased as

$$pred_k(s_k) \geq (1-\eta_k)\|F(x_k)\|, \tag{2.5}$$

$$ared_k(s_k) \geq t(1-\eta_k)\|F(x_k)\|. \tag{2.6}$$

Note in particular that if s_k is any step that reduces the norm of the local linear model and if η_k is chosen so that (2.1)/(2.5) is satisfied with equality, then (2.6) is equivalent to (2.3). Thus (2.1) and (2.2) can be regarded as an extension of tests of the form (2.3) to the inexact Newton context.

Less closely related tests are the two *Goldstein–Armijo conditions* derived from work in [10] and [1]. Typically only the first (the "alpha condition") is implemented in practice (see, e.g., Dennis and Schnabel [5, Chap. 6]); this is

$$f(x_k+s_k) \leq f(x_k) + \alpha \nabla f(x_k)^T s_k, \tag{2.7}$$

where $f \equiv \frac{1}{2}\|F\|_2^2$ and $0 < \alpha < 1$. The following proposition shows that when $\|\cdot\| = \|\cdot\|_2$, the alpha condition implies (2.2) for steps satisfying (2.1). Consequently, algorithms that generate steps satisfying (2.1) and the alpha condition, such as those of Brown and Saad [3], can be regarded as special cases of Algorithm GIN.

PROPOSITION 2.1. *Let* $\|\cdot\| = \|\cdot\|_2$ *and* $f \equiv \frac{1}{2}\|F\|_2^2$. *If* (2.1) *and* (2.7) *hold, then* (2.2) *also holds with* $t = \alpha$.

Proof. It follows from (2.7) that

$$\|F(x_k+s_k)\|_2^2 \leq \|F(x_k)\|_2^2 + 2\alpha F(x_k)^T F'(x_k)\, s_k.$$

Writing $r \equiv F(x_k) + F'(x_k)\, s_k$, we obtain from (2.1) that

$$\begin{aligned} F(x_k)^T F'(x_k)\, s_k &= F(x_k)^T[-F(x_k)+r] \\ &= -\|F(x_k)\|_2^2 + F(x_k)^T r \\ &\leq -(1-\eta_k)\|F(x_k)\|_2^2, \end{aligned}$$

whence

$$\|F(x_k+s_k)\|_2^2 \leq [1-2\alpha(1-\eta_k)]\|F(x_k)\|_2^2.$$

Since the left-hand side is nonnegative, we must have $2\alpha(1-\eta_k) \leq 1$; since $\sqrt{1-\epsilon} \leq 1-\epsilon/2$ whenever $|\epsilon| \leq 1$, we also have

$$\|F(x_k+s_k)\|_2 \leq [1-\alpha(1-\eta_k)]\,\|F(x_k)\|_2. \qquad \square$$

3. Convergence theorems. In this section, we develop a theoretical foundation for Algorithm GIN. This provides a basis for the analysis of all of the algorithms considered in the sequel. We first address breakdown in Lemma 3.1 and Corollary 3.2 below.

LEMMA 3.1. *Let x and $t \in (0,1)$ be given and assume that there exists an $\bar{s}$ that satisfies $\|F(x)+F'(x)\bar{s}\| < \|F(x)\|$. Then there exists $\eta_{\min} \in [0,1)$ such that, for any $\eta \in [\eta_{\min}, 1)$, there is an s satisfying*

$$\|F(x)+F'(x)s\| \le \eta\|F(x)\| \quad \text{and} \quad \|F(x+s)\| \le [1-t(1-\eta)]\,\|F(x)\|.$$

Proof. Clearly $F(x) \neq 0$ and $\bar{s} \neq 0$. Set

$$\bar{\eta} \equiv \frac{\|F(x)+F'(x)\bar{s}\|}{\|F(x)\|},$$

$$\epsilon \equiv \frac{(1-t)(1-\bar{\eta})\|F(x)\|}{\|\bar{s}\|},$$

$$\eta_{\min} \equiv \max\left\{\bar{\eta},\ 1-\frac{(1-\bar{\eta})\delta}{\|\bar{s}\|}\right\},$$

where $\delta > 0$ is sufficiently small that

$$\|F(x+s)-F(x)-F'(x)s\| \le \epsilon\|s\|$$

whenever $\|s\| \le \delta$. Such a δ exists by Lemma 1.2.

For any $\eta \in [\eta_{\min}, 1)$, let $s \equiv \frac{1-\eta}{1-\bar{\eta}}\bar{s}$. Then

$$\begin{aligned}\|F(x)+F'(x)s\| &\le \frac{\eta-\bar{\eta}}{1-\bar{\eta}}\|F(x)\| + \frac{1-\eta}{1-\bar{\eta}}\|F(x)+F'(x)\bar{s}\| \\ &= \frac{\eta-\bar{\eta}}{1-\bar{\eta}}\|F(x)\| + \frac{1-\eta}{1-\bar{\eta}}\bar{\eta}\|F(x)\| \\ &= \eta\|F(x)\|,\end{aligned}$$

and, since

$$\|s\| = \frac{1-\eta}{1-\bar{\eta}}\|\bar{s}\| \le \frac{1-\eta_{\min}}{1-\bar{\eta}}\|\bar{s}\| \le \delta,$$

it follows that

$$\begin{aligned}\|F(x+s)\| &\le \|F(x+s)-F(x)-F'(x)s\| + \|F(x)+F'(x)s\| \\ &\le \epsilon\cdot\frac{1-\eta}{1-\bar{\eta}}\|\bar{s}\| + \eta\|F(x)\| \\ &= (1-t)(1-\eta)\,\|F(x)\| + \eta\|F(x)\| \\ &= [1-t(1-\eta)]\,\|F(x)\|. \qquad \square\end{aligned}$$

COROLLARY 3.2. *Algorithm* GIN *breaks down at some x_k if and only if there is no inexact Newton step from x_k, i.e., if and only if $F(x_k) \neq 0$ and x_k is a stationary point of $\|F\|$.*

The next theorem is used in the proof of the global convergence theorem that follows and is also of independent interest.

THEOREM 3.3. *Assume that $\{x_k\}$ is a sequence such that $F(x_k) \to 0$ and, for each k,*

$$\|F(x_k)+F'(x_k)s_k\| \le \eta\|F(x_k)\| \quad \text{and} \quad \|F(x_{k+1})\| \le \|F(x_k)\|,$$

where $s_k \equiv x_{k+1} - x_k$ *and* η *is independent of* k. *If* x_* *is a limit point of* $\{x_k\}$ *such that* $F'(x_*)$ *is invertible, then* $F(x_*) = 0$ *and* $x_k \to x_*$.

Proof. Clearly $F(x_*) = 0$. Set $K \equiv \|F'(x_*)^{-1}\|$, and let $\delta > 0$ be sufficiently small that $F'(y)^{-1}$ exists and

$$\|F'(y)^{-1}\| \le 2K,$$

$$\|F(y) - F(x_*) - F'(x_*)(y - x_*)\| \le \frac{1}{2K}\|y - x_*\|,$$

whenever $y \in N_\delta(x_*)$. Such a δ exists by Lemmas 1.1 and 1.2.

If $y \in N_\delta(x_*)$, then

$$\begin{aligned}\|F(y)\| &\ge \|F'(x_*)(y - x_*)\| - \|F(y) - F(x_*) - F'(x_*)(y - x_*)\| \\ &\ge \frac{1}{\|F'(x_*)^{-1}\|}\|y - x_*\| - \frac{1}{2K}\|y - x_*\| \\ &= \frac{1}{2K}\|y - x_*\|,\end{aligned}$$

so that

$$\|y - x_*\| \le 2K\|F(y)\| \tag{3.1}$$

whenever $y \in N_\delta(x_*)$.

Let $\epsilon \in (0, \delta/4)$ be given. Since x_* is a limit point of $\{x_k\}$ and $F(x_*) = 0$, there is a k sufficiently large that

$$x_k \in S_\epsilon \equiv \{y \mid \|y - x_*\| < \delta/2 \text{ and } \|F(y)\| < \epsilon/[K(1+\eta)]\}.$$

Then

$$\begin{aligned}\|s_k\| &= \|F'(x_k)^{-1}[-F(x_k) + \{F(x_k) + F'(x_k)\, s_k\}]\| \\ &\le \|F'(x_k)^{-1}\|(\|F(x_k)\| + \|F(x_k) + F'(x_k)\, s_k\|) \\ &\le 2K(1+\eta)\|F(x_k)\| \\ &< 2\epsilon \\ &< \frac{\delta}{2},\end{aligned}$$

and so

$$\|x_{k+1} - x_*\| \le \|x_k - x_*\| + \|s_k\| < \delta.$$

Since

$$\|F(x_{k+1})\| \le \|F(x_k)\| < \epsilon/[K(1+\eta)]$$

and, by (3.1),

$$\|x_{k+1} - x_*\| \le 2K\|F(x_{k+1})\| < 2K \cdot \epsilon/[K(1+\eta)] < \delta/2,$$

it follows that $x_{k+1} \in S_\epsilon$. We conclude that $x_k \in S_\epsilon \subseteq N_\delta(x_*)$ for all sufficiently large k, and, since $\|F(x_k)\| \to 0$, it follows from (3.1) that $x_k \to x_*$. $\square$

We now establish the basic global convergence theorem for Algorithm GIN.

THEOREM 3.4 (global convergence of algorithm GIN). *Assume that Algorithm* GIN *does not break down. If* $\sum_{k\geq 0}(1-\eta_k)$ *is divergent, then* $F(x_k) \to 0$. *If, in addition,* x_* *is a limit point of* $\{x_k\}$ *such that* $F'(x_*)$ *is invertible, then* $F(x_*) = 0$ *and* $x_k \to x_*$.

Proof. By (2.2),

$$\begin{aligned}\|F(x_k)\| &\leq [1-t(1-\eta_{k-1})]\,\|F(x_{k-1})\| \\ &\leq \|F(x_0)\| \prod_{0\leq j<k} [1-t(1-\eta_j)] \\ &\leq \|F(x_0)\| \exp\left[-t\sum_{0\leq j<k}(1-\eta_j)\right].\end{aligned}$$

Since $t > 0$ and $1-\eta_j \geq 0$, the divergence of $\sum_{k\geq 0}(1-\eta_k)$ implies $F(x_k) \to 0$. The remainder of the theorem follows from Theorem 3.3 with $\eta = 1$. □

With (2.2), the divergence of $\sum_{k\geq 0}(1-\eta_k)$ implies sufficient cumulative decrease in $\|F\|$ over all steps to ensure that $F(x_k) \to 0$. In applications of Theorem 3.4 in the sequel, conditions are given under which this divergence holds implicitly; it is not necessary to assume it explicitly. In general, however, it may not be possible to determine the η_k's so that $\sum_{k\geq 0}(1-\eta_k)$ is divergent, even though Algorithm GIN does not break down. For example, define $F(x) \equiv 1+\exp(-x^2)$ for $x \in \mathbf{R}^1$. If $x_0 \neq 0$, then the full Newton step is always a well-defined inexact Newton step from the current x_k and, for any $t \in (0,1)$, admissible η_k and s_k exist by Lemma 3.1. However, $\sum_{k\geq 0}(1-\eta_k)$ cannot be divergent because $\{F(x_k)\}$ cannot converge to zero.

Note that (2.1) is not needed to obtain the weak result that $F(x_k) \to 0$; only (2.2) is used. However, (2.2) alone implies nothing further about the convergence of $\{x_k\}$. Indeed, for $F: \mathbf{R}^1 \to \mathbf{R}^1$ given by $F(x) \equiv x^2 - 1$, the solutions are ± 1; if $\{x_k\}$ converges to $+1$ and satisfies (2.2) for some $\{\eta_k\}$, then $\{(-1)^k x_k\}$ also satisfies (2.2) and has both $+1$ and -1 as limit points. To show that $x_k \to x_*$, we apply Theorem 3.3, which in effect uses (2.1) to ensure that iterates remain near solutions at which F' is invertible, thereby precluding the sort of behavior seen in this example.

Another way to state Theorem 3.4 is that if Algorithm GIN does not break down and $\sum_{k\geq 0}(1-\eta_k)$ is divergent, then $F(x_k) \to 0$ and one of the following holds:

1. $\|x_k\| \to \infty$, i.e., $\{x_k\}$ has no limit points;
2. $\{x_k\}$ has one or more limit points, and F' is singular at each of them;
3. $\{x_k\}$ converges to a solution x_* at which F' is invertible.

These alternatives allow useful corollaries to be drawn. For example, if F' is invertible on $\{x \mid \|F(x)\| \leq \|F(x_0)\|\}$ and this level set is bounded, then alternative 3 must hold and the ultimate rate of convergence is determined by the values of η_k for large k as in the local theory of Dembo, Eisenstat, and Steihaug [4].

Alternative 1 can occur. Indeed, if $F(x) \equiv \exp(-x^2)$ for $x \in \mathbf{R}^1$ and if $x_0 \neq 0$ and $t = \frac{1}{2}$, then full Newton steps are admissible in Algorithm GIN and $|x_k| \to \infty$.

Alternative 2 can also occur, and there can clearly be convergence to a point at which F' is singular. There can also be more than one limit point of $\{x_k\}$. For example, define $F(x) \equiv (x^{(1)}, 0)^T$ for $x = (x^{(1)}, x^{(2)})^T \in \mathbf{R}^2$. For each k, we take $\eta_k = \frac{1}{2}$ and $s_k = (-x_k^{(1)}/2, s_k^{(2)})^T$, where $x_k = (x_k^{(1)}, x_k^{(2)})^T$ and $s_k^{(2)}$ is arbitrary. Then (2.1) and (2.2) hold for any $t \in (0,1)$. Furthermore, the limit points of $\{x_k\}$ constitute a subset of the solution set $\{(0, x^{(2)})^T\}$, and, since each $s_k^{(2)}$ is arbitrary, this subset can be made to contain any given subset of the solution set.

The next theorem is complementary to Theorem 3.3 and is useful in the sequel.

THEOREM 3.5. *Assume that Algorithm* GIN *does not break down. If x_* is a limit point of $\{x_k\}$ such that there exists a Γ independent of k for which*

$$\|s_k\| \leq \Gamma(1 - \eta_k)\|F(x_k)\| \tag{3.2}$$

whenever x_k is sufficiently near x_* and k is sufficiently large, then $x_k \to x_*$.

Proof. Suppose that $x_k \not\to x_*$. Let $\delta > 0$ be such that there exist infinitely many k for which $x_k \notin N_\delta(x_*)$ and sufficiently small that (3.2) holds whenever $x_k \in N_\delta(x_*)$ and k is sufficiently large.

Since x_* is a limit point of $\{x_k\}$, there exist $\{k_j\}$ and $\{\ell_j\}$ such that, for each j,

$$\begin{aligned}
&x_{k_j} \in N_{\delta/j}(x_*),\\
&x_{k_j+i} \in N_\delta(x_*), \qquad i = 0, \ldots, l_j - 1,\\
&x_{k_j+l_j} \notin N_\delta(x_*),\\
&k_j + l_j < k_{j+1}.
\end{aligned}$$

Then for j sufficiently large,

$$\begin{aligned}
\delta/2 &\leq \|x_{k_j+l_j} - x_{k_j}\|\\
&\leq \sum_{k=k_j}^{k_j+l_j-1} \|s_k\|\\
&\leq \sum_{k=k_j}^{k_j+l_j-1} \Gamma(1-\eta_k)\|F(x_k)\|\\
&\leq \sum_{k=k_j}^{k_j+l_j-1} \frac{\Gamma}{t}\{\|F(x_k)\| - \|F(x_{k+1})\|\}\\
&= \frac{\Gamma}{t}\left\{\|F(x_{k_j})\| - \|F(x_{k_j+l_j})\|\right\}\\
&\leq \frac{\Gamma}{t}\left\{\|F(x_{k_j})\| - \|F(x_{k_{j+1}})\|\right\}.
\end{aligned}$$

But the last right-hand side converges to zero since $x_{k_j} \to x_*$; hence, this inequality cannot hold for large j. □

Theorems 3.4 and 3.5 are still valid if we allow $\eta_k \in [0, 1]$ in Algorithm GIN. The resulting more general statements can be adapted to provide global convergence results for *any* sequence $\{x_k\}$ such that each $s_k \equiv x_{k+1} - x_k$ does not increase the norm of the local linear model of F, i.e., such that $pred_k(s_k) \geq 0$ for each k. Indeed, for each k, we can *define* $\eta_k \in [0, 1]$ by

$$\eta_k \equiv \begin{cases} \|F(x_k) + F'(x_k)\, s_k\|/\|F(x_k)\| & \text{if } F(x_k) \neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Then (2.1)/(2.5) holds with equality, and η_k and s_k are acceptable in Algorithm GIN if and only if (2.3) holds, i.e, $ared_k(s_k) \geq t \cdot pred_k(s_k)$. Moreover, introducing

$$relpred_k(s_k) \equiv \begin{cases} pred_k(s_k)/\|F(x_k)\| & \text{if } F(x_k) \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

we have

$$\sum_{k\geq 0}(1-\eta_k) = \sum_{k\geq 0} relpred_k(s_k).$$

Then Theorem 3.4 can be restated as follows.

COROLLARY 3.6. *Assume that $\{x_k\}$ is such that $pred_k(s_k) \geq 0$ and $ared_k(s_k) \geq t \cdot pred_k(s_k)$ for each k, where $t \in (0,1)$ is independent of k. If $\sum_{k\geq 0} relpred_k(s_k)$ is divergent, then $F(x_k) \to 0$. If, in addition, x_* is a limit point of $\{x_k\}$ such that $F'(x_*)$ is invertible, then $F(x_*) = 0$ and $x_k \to x_*$.*

This result can be viewed as an analogue of the results of Wolfe [18], [19] on sequences that proceed through descent-direction steps for $f: \mathbf{R}^n \to \mathbf{R}^1$ that satisfy the Goldstein–Armijo conditions. Theorem 3.5 can be restated in a similar manner.

COROLLARY 3.7. *Assume that $\{x_k\}$ is such that $pred_k(s_k) \geq 0$ and $ared_k(s_k) \geq t \cdot pred_k(s_k)$ for each k, where $t \in (0,1)$ is independent of k. If x_* is a limit point of $\{x_k\}$ such that there exists a Γ independent of k for which $\|s_k\| \leq \Gamma \cdot pred_k(s_k)$ whenever x_k is sufficiently near x_* and k is sufficiently large, then $x_k \to x_*$.*

4. Applications. Although the developments in §§2 and 3 are intended primarily to provide a groundwork for §§5–8, they also have broader applicability to globalizations of Algorithm N and related methods, which we now illustrate. Material in this section is not used in later sections.

Application 1. Trust region methods. There are a variety of trust region methods for unconstrained optimization and general nonlinear equations; see, e.g., Dennis and Schnabel [5, §6.4], El Hallabi [7], El Hallabi and Tapia [8], Moré and Sorensen [12], Shultz, Schnabel, and Byrd [16], and the references therein. Here, we show how the developments in §§2 and 3 can be brought to bear on these methods without giving an exhaustive treatment. The algorithm below is meant to reflect the essential features of several known trust region algorithms; it is especially close to Algorithm 4.2 of Moré and Sorensen [12].

ALGORITHM TR (trust region method).
> LET x_0, $\bar{\delta}_0 > 0$, $0 < t \leq u < 1$, AND $0 < \theta_{\min} < \theta_{\max} < 1$ BE GIVEN.
> FOR $k = 0$ STEP 1 UNTIL ∞ DO:
>> SET $\delta_k = \bar{\delta}_k$ AND CHOOSE $s_k \in \arg\min_{\|s\|\leq\delta_k} \|F(x_k) + F'(x_k)\, s\|$.
>> WHILE $ared_k(s_k) < t \cdot pred_k(s_k)$ DO:
>>> CHOOSE $\theta \in [\theta_{\min}, \theta_{\max}]$.
>>> UPDATE $\delta_k \leftarrow \theta\delta_k$ AND CHOOSE
>>>> $s_k \in \arg\min_{\|s\|\leq\delta_k} \|F(x_k) + F'(x_k)\, s\|$.
>> SET $x_{k+1} = x_k + s_k$.
>> IF $ared_k(s_k) \geq u \cdot pred_k(s_k)$ CHOOSE $\bar{\delta}_{k+1} \geq \delta_k$;
>>> ELSE CHOOSE $\bar{\delta}_{k+1} \geq \theta_{\min}\delta_k$.

Algorithm TR and the analysis below could easily be modified to allow steps that are only approximate minimizers of the local linear model norm within trust regions, the use of a second norm in determining δ_k-balls as in El Hallabi [7] and El Hallabi and Tapia [8], additional refinements in the determination of trust region radii, etc. It can easily be shown that Algorithm TR does not break down if, for each k, either $F'(x_k)$ is invertible or x_k is not a stationary point of $\|F\|$.

To develop a convergence analysis for Algorithm TR, we first use Corollary 3.7 to obtain the following lemma.

LEMMA 4.1. *Assume that Algorithm* TR *does not break down. Suppose that x_* is a limit point of $\{x_k\}$ such that there exists a Γ independent of k for which*

$$\|s_k\| \le \Gamma \cdot pred_k(s_k) \tag{4.1}$$

whenever x_k is sufficiently near x_* and k is sufficiently large. Then $x_k \to x_*$ and $\liminf_{k\to\infty} \delta_k > 0$.

Remark. One can also conclude that x_* is a stationary point, but this is not needed here.

Proof. It is clear that $\{x_k\}$ satisfies the hypotheses of Corollary 3.7, and it follows immediately that $x_k \to x_*$. Choose $\delta > 0$ such that (4.1) holds whenever $x_k \in N_\delta(x_*)$ and k is sufficiently large and also such that

$$\|F(y) - F(x) - F'(x)\,(y-x)\| \le \frac{(1-u)}{\Gamma}\|y-x\| \tag{4.2}$$

whenever $x, y \in N_\delta(x_*)$. Let k_0 be such that if $k \ge k_0$, then $x_k \in N_{\delta/2}(x_*)$ and (4.1) holds.

We claim that if $k \ge k_0$, then the while-loop in Algorithm TR terminates with

$$\delta_k \ge \min\left\{\bar\delta_{k_0}, \theta_{\min}\delta/2\right\},$$

whence $\liminf_{k\to\infty}\delta_k \ge \min\left\{\bar\delta_{k_0}, \theta_{\min}\delta/2\right\} > 0$. To show this, we first note that if $k \ge k_0$ and if s_k is a trial step for which $\|s_k\| \le \delta/2$, then (4.2) and (4.1) give

$$\begin{aligned} ared_k(s_k) &\equiv \|F(x_k)\| - \|F(x_k+s_k)\| \\ &\ge \|F(x_k)\| - \|F(x_k) + F'(x_k)\,s_k\| - \|F(x_k+s_k) - F(x_k) - F'(x_k)\,s_k\| \\ &\ge pred_k(s_k) - \frac{(1-u)}{\Gamma}\|s_k\| \\ &\ge u \cdot pred_k(s_k). \end{aligned}$$

It follows from this that the while-loop terminates with

$$\delta_k \ge \min\left\{\bar\delta_k, \theta_{\min}\delta/2\right\}. \tag{4.3}$$

Indeed, if the while-loop does not reduce δ_k, then $\delta_k = \bar\delta_k$ on termination; whereas, if the while-loop reduces δ_k at least once, then the penultimate value is at least $\delta/2$, whence $\delta_k \ge \theta_{\min}\delta/2$ on termination. Furthermore, if $\delta_k \le \delta/2$ on termination, then $\bar\delta_{k+1} \ge \delta_k$; whereas, if $\delta_k > \delta/2$ on termination, then $\bar\delta_{k+1} \ge \theta_{\min}\delta_k > \theta_{\min}\delta/2$. Thus

$$\bar\delta_{k+1} \ge \min\left\{\bar\delta_k, \theta_{\min}\delta/2\right\}. \tag{4.4}$$

The claim follows by induction from (4.3) and (4.4). □

The next two lemmas allow us to apply Lemma 4.1 in the proof of Theorem 4.4, which is our main result.

LEMMA 4.2. *If x_* is such that $F'(x_*)$ is invertible, then there exist Γ and $\epsilon_* > 0$ such that, for any $\delta > 0$,*

$$s \in \arg \min_{\|\bar{s}\| \le \delta} \|F(x) + F'(x)\,\bar{s}\| \tag{4.5}$$

satisfies

$$\|s\| \le \Gamma\{\|F(x)\| - \|F(x) + F'(x)\,s\|\} \tag{4.6}$$

whenever $x \in N_{\epsilon_*}(x_*)$.

Proof. Set $K \equiv \|F'(x_*)^{-1}\|$ and let $\epsilon_* > 0$ be sufficiently small that $F'(x)$ is invertible and $\|F'(x)^{-1}\| \le 2K$ whenever $x \in N_{\epsilon_*}(x_*)$. Suppose that $x \in N_{\epsilon_*}(x_*)$ and s is given by (4.5) for an arbitrary $\delta > 0$. Denote $s^{\mathrm{N}} \equiv -F'(x)^{-1}F(x)$, and note that $\|s\| \le \|s^{\mathrm{N}}\|$ since s^{N} is the unique global minimizer of the norm of the local linear model. If $s^{\mathrm{N}} = 0$, then $s = 0$ and (4.6) holds trivially for any Γ. If $s^{\mathrm{N}} \ne 0$, then

$$\begin{aligned}
\|F(x)\| - \|F(x) + F'(x)\,s\| &\ge \|F(x)\| - \left\|F(x) + F'(x)\,\frac{\|s\|}{\|s^{\mathrm{N}}\|}s^{\mathrm{N}}\right\| \\
&= \frac{\|s\|}{\|s^{\mathrm{N}}\|}\|F(x)\| \\
&\ge \frac{\|s\|}{2K},
\end{aligned}$$

and (4.6) holds with $\Gamma \equiv 2K$. Thus (4.6) holds with $\Gamma \equiv 2K$ $\forall x \in N_{\epsilon_*}(x_*)$. □

LEMMA 4.3. *If x_* is not a stationary point of $\|F\|$, then there exist Γ, $\delta_* > 0$, and $\epsilon_* > 0$ such that s given by (4.5) satisfies (4.6) whenever $x \in N_{\epsilon_*}(x_*)$ and $0 < \delta \le \delta_*$.*

Proof. Let $\epsilon > 0$ be such that if $x \in N_\epsilon(x_*)$, then $\|F(x)\| \ge \frac{1}{2}\|F(x_*)\|$. Let s_* be such that $\|F(x_*) + F'(x_*)\,s_*\| < \|F(x_*)\|$. Choose η_* such that

$$\|F(x_*) + F'(x_*)\,s_*\|/\|F(x_*)\| < \eta_* < 1.$$

Since F and F' are continuous, there exists $\epsilon_* \in (0, \epsilon]$ such that

$$\|F(x) + F'(x)\,s_*\| \le \eta_*\|F(x)\|$$

whenever $x \in N_{\epsilon_*}(x_*)$. Choose $\delta_* \in (0, \|s_*\|)$. Suppose that $x \in N_{\epsilon_*}(x_*)$ and $0 < \delta \le \delta_*$. For s given by (4.5), we have

$$\begin{aligned}
\|F(x)\| - \|F(x) + F'(x)\,s\| &\ge \|F(x)\| - \left\|F(x) + F'(x)\,\frac{\|s\|}{\|s_*\|}s_*\right\| \\
&\ge \|F(x)\| - \left(1 - \frac{\|s\|}{\|s_*\|}\right)\|F(x)\| \\
&\qquad - \frac{\|s\|}{\|s_*\|}\|F(x) + F'(x)\,s_*\| \\
&\ge \frac{(1-\eta_*)\|F(x)\|}{\|s_*\|}\|s\| \\
&\ge \frac{(1-\eta_*)\|F(x_*)\|}{2\|s_*\|}\|s\|,
\end{aligned}$$

and (4.6) holds with $\Gamma \equiv 2\|s_*\|/[(1-\eta_*)\|F(x_*)\|]$. □

THEOREM 4.4. *Assume that Algorithm* TR *does not break down. Then every limit point of* $\{x_k\}$ *is a stationary point of* $\|F\|$. *If* x_* *is a limit point of* $\{x_k\}$ *such that* $F'(x_*)$ *is invertible, then* $F(x_*) = 0$ *and* $x_k \to x_*$; *furthermore,* $s_k = -F'(x_k)^{-1}F(x_k)$, *the full Newton step, whenever* k *is sufficiently large.*

Proof. Suppose that x_* is a limit point of $\{x_k\}$ that is not a stationary point of $\|F\|$.

We claim that, for any $\delta > 0$, there exists an $\epsilon > 0$ such that if $x_k \in N_\epsilon(x_*)$ and k is sufficiently large, then $\delta_k \leq \delta$. Otherwise, there would exist a $\delta > 0$ and $\{x_{k_j}\} \subseteq \{x_k\}$ such that $x_{k_j} \to x_*$ and $\delta_{k_j} > \delta$ for each j. Then

$$\begin{aligned}
0 &= \lim_{j\to\infty} \{\|F(x_{k_j})\| - \|F(x_{k_{j+1}})\|\} \\
&\geq \liminf_{j\to\infty} \{\|F(x_{k_j})\| - \|F(x_{k_j+1})\|\} \\
&= \liminf_{j\to\infty} ared_{k_j}(s_{k_j}) \\
&\geq t \cdot \liminf_{j\to\infty} pred_{k_j}(s_{k_j}) \\
&= t \cdot \liminf_{j\to\infty} \left\{ \|F(x_{k_j})\| - \min_{\|s\|\leq\delta_{k_j}} \|F(x_{k_j}) + F'(x_{k_j})\, s\| \right\} \\
&\geq t \cdot \lim_{j\to\infty} \left\{ \|F(x_{k_j})\| - \min_{\|s\|\leq\delta} \|F(x_{k_j}) + F'(x_{k_j})\, s\| \right\} \\
&= t \cdot \left\{ \|F(x_*)\| - \min_{\|s\|\leq\delta} \|F(x_*) + F'(x_*)\, s\| \right\}.
\end{aligned}$$

But the last right-hand side must be positive since x_* is not a stationary point.

Now let Γ, δ_*, and ϵ_* be as in Lemma 4.3. By the above claim, there exists $\epsilon \in (0, \epsilon_*]$ such that if $x_k \in N_\epsilon(x_*)$ and k is sufficiently large, then $\delta_k \leq \delta_*$. By Lemma 4.3, (4.1) holds for Γ independent of k whenever $x_k \in N_\epsilon(x_*)$ and k is sufficiently large. Then Lemma 4.1 implies that $x_k \to x_*$ and $\liminf_{k\to\infty} \delta_k > 0$. But since the claim implies that $\delta_k \to 0$, this is a contradiction. Hence, x_* must be a stationary point.

Suppose that x_* is a limit point of $\{x_k\}$ such that $F'(x_*)$ is invertible. Since x_* must be a stationary point, we must have $F(x_*) = 0$. It follows from Lemma 4.2 that there exists a Γ independent of k for which (4.1) holds whenever x_k is sufficiently near x_*. Then Lemma 4.1 implies that $x_k \to x_*$ and there exists a $\delta > 0$ such that $\delta_k \geq \delta$ for sufficiently large k. Since $x_k \to x_*$ and $F(x_k) \to F(x_*) = 0$, we have that $\|F'(x_k)^{-1}F(x_k)\| \leq \delta \leq \delta_k$ and, therefore, $s_k = -F'(x_k)^{-1}F(x_k)$ whenever k is sufficiently large. □

Application 2. Global approximate Newton methods. The global approximate Newton method of Bank and Rose [2] can be written as follows.

ALGORITHM GAN (global approximate Newton method [2]).

LET x_0 BE GIVEN; DETERMINE $K_0 \geq 0$.
FOR $k = 0$ STEP 1 UNTIL ∞ DO:
  SOLVE $M_k\,\bar{s}_k = -F(x_k)$, WHERE $M_k \approx F'(x_k)$.
  CHOOSE $K_k \in [0, K_0]$.
  SET $s_k = \tau_k \bar{s}_k$, WHERE $\tau_k \equiv 1/(1 + K_k\|F(x_k)\|)$.
  SET $x_{k+1} = x_k + s_k$.

Here, we show that Algorithm GAN is a special case of Algorithm GIN and that the global convergence result below of Bank and Rose [2, §2] follows from Theorem 3.4 under the assumptions in [2, §2], viz., the following:

(1) $L(x_0) \equiv \{x| \, \|F(x)\| \leq \|F(x_0)\|\}$ is bounded;
(2) F' is invertible on $L(x_0)$, each M_k is invertible, and $\|M_k^{-1}\| \leq \kappa$ for all $k \geq 0$;
(3) $\|F'(y) - F'(x)\| \leq \gamma\|y - x\|$ for x, $y \in \{u| \, \|u\| \leq \sup_{v\in L(x_0)} \|v\| + \kappa\|F(x_0)\|\}$;
(4) $F(x_k) \neq 0$ and $\bar{\eta}_k \equiv \|F(x_k) + F'(x_k)\bar{s}_k\|/\|F(x_k)\| \leq \bar{\eta}_0 < 1$ for all $k \geq 0$;
(5) For $t \in (0, 1 - \bar{\eta}_0)$, $K_k \geq (\kappa^2\gamma/2)(1 - \bar{\eta}_k - t)^{-1} - \|F(x_k)\|^{-1}$ for all $k \geq 0$.

PROPOSITION 4.5 (cf. conclusion (i) of [2, Theorem 1, p. 285]). *Under assumptions* (1)–(5) *above, there exists an* x_* *such that* $F(x_*) = 0$ *and* $x_k \to x_*$.

Proof. Setting $\eta_k \equiv (1 - \tau_k) + \tau_k\bar{\eta}_k$ for each k, we have that $\eta_k \in [0, 1)$ and

$$\begin{aligned}\|F(x_k) + F'(x_k)s_k\| &= \|(1 - \tau_k)F(x_k) + \tau_k[F(x_k) + F'(x_k)\bar{s}_k]\| \\ &\leq \eta_k\|F(x_k)\|,\end{aligned}$$

i.e., (2.1) holds. By Bank and Rose [2, (2.18), p. 283], we also have

$$\|F(x_k + s_k)\| \leq (1 - t\tau_k)\|F(x_k)\|.$$

Since

$$1 - t\tau_k \leq 1 - t\tau_k(1 - \bar{\eta}_k) = 1 - t(1 - \eta_k),$$

this gives

$$\|F(x_k + s_k)\| \leq [1 - t(1 - \eta_k)]\,\|F(x_k)\|,$$

i.e., (2.2) holds. Thus Algorithm GAN is a special case of Algorithm GIN.

To complete the proof, we note that

$$1 - \eta_k = \tau_k(1 - \bar{\eta}_k) \geq \tau_k(1 - \bar{\eta}_0)$$

and

$$\tau_k \equiv \frac{1}{1 + K_k\|F(x_k)\|} \geq \frac{1}{1 + K_0\|F(x_0)\|};$$

hence, $\sum_{k\geq 0}(1 - \eta_k)$ is divergent. Moreover, assumptions (1) and (2) imply that $\{x_k\}$ has a limit point x_* such that $F'(x_*)$ is invertible. Then Theorem 3.4 implies that $F(x_*) = 0$ and $x_k \to x_*$. □

5. Two paradigm methods. We begin now to develop methods that test initial inexact Newton steps and, if they are unsatisfactory, determine new steps as necessary until steps satisfying (2.1) and (2.2) are found. Our ultimate goal is to outline particular algorithms in §§6–8 that can be implemented as practical algorithms. To simplify their treatment, we formulate and analyze in this section two paradigm methods that incorporate their basic features.

At each iteration of these paradigm methods, an initial inexact Newton step at a specified level is tried and, if it proves unsatisfactory, then inexact Newton steps at higher levels are tried until a satisfactory step is found. Accordingly, we assume that, at the kth iteration of each method, there is a means of specifying *some* inexact

Newton step from x_k for each η in $[\bar{\eta}_k, 1]$, where $\bar{\eta}_k \in [0, 1)$ is the initial inexact Newton level. We express this by assuming that, for each k, there is a curve σ_k satisfying

$$\|F(x_k) + F'(x_k)\,\sigma_k(\eta)\| \leq \eta\|F(x_k)\|, \qquad \bar{\eta}_k \leq \eta \leq 1. \tag{5.1}$$

For now, σ_k is not required to have any properties other than (5.1).

We call the first method a *minimum reduction method*: At each iteration, the initial inexact Newton step is required to achieve at least a prescribed minimum reduction of the norm of the local linear model.

ALGORITHM MR (minimum reduction method).

LET x_0, $\eta_{\max} \in [0, 1)$, $t \in (0, 1)$, AND $0 < \theta_{\min} < \theta_{\max} < 1$ BE GIVEN.
FOR $k = 0$ STEP 1 UNTIL ∞ DO:
  CHOOSE $\bar{\eta}_k \in [0, \eta_{\max}]$.
  DETERMINE σ_k SUCH THAT (5.1) HOLDS; SET $\eta_k = \bar{\eta}_k$.
  WHILE $\|F(x_k + \sigma_k(\eta_k))\| > [1 - t(1 - \eta_k)]\,\|F(x_k)\|$ DO:
    CHOOSE $\theta \in [\theta_{\min}, \theta_{\max}]$.
    UPDATE $\eta_k \leftarrow 1 - \theta(1 - \eta_k)$.
  SET $x_{k+1} = x_k + \sigma_k(\eta_k)$.

We call the second method a *trust level method*: At each iteration, the initial reduction required in the local linear model norm reflects a level of trust in the model based on behavior observed at previous iterations. The term "trust level" is intended both to evoke and to contrast this method with trust *region* methods, in which steps are determined within *regions* of trust of local linear models of F. The structure of this method has obvious parallels with that of Algorithm TR in §4.

ALGORITHM TL (trust level method).

LET x_0, $\bar{\eta}_0 \in [0, 1)$, $0 < t \leq u < 1$, AND $0 < \theta_{\min} < \theta_{\max} < 1$ BE GIVEN.
FOR $k = 0$ STEP 1 UNTIL ∞ DO:
  DETERMINE σ_k SUCH THAT (5.1) HOLDS; SET $\eta_k = \bar{\eta}_k$.
  WHILE $\|F(x_k + \sigma_k(\eta_k))\| > [1 - t(1 - \eta_k)]\,\|F(x_k)\|$ DO:
    CHOOSE $\theta \in [\theta_{\min}, \theta_{\max}]$.
    UPDATE $\eta_k \leftarrow 1 - \theta(1 - \eta_k)$.
  SET $x_{k+1} = x_k + \sigma_k(\eta_k)$.
  IF $\|F(x_{k+1})\| \leq [1 - u(1 - \eta_k)]\,\|F(x_k)\|$ CHOOSE $\bar{\eta}_{k+1} \in [0, \eta_k]$;
   ELSE CHOOSE $\bar{\eta}_{k+1} \in [0, 1 - \theta_{\min}(1 - \eta_k)]$.

The restriction $\bar{\eta}_k \in [0, \eta_{\max}]$ in Algorithm MR should not be a serious practical restriction since $\eta_{\max}$ can be taken arbitrarily near one. The restriction $\theta \in [\theta_{\min}, \theta_{\max}]$ in the while-loop in both algorithms is reflective of typical "safeguarding" practices; see, e.g., Dennis and Schnabel [5, §§6.3.2 and 6.4.3]. This still allows a great deal of flexibility in the choice of θ.

Whether either algorithm breaks down at the kth iteration depends on both the existence and the nature of σ_k satisfying (5.1). *Some* such σ_k exists if and only if there exists *some* inexact Newton step at the level $\bar{\eta}_k$. Indeed, the existence of such an inexact Newton step is clearly necessary, and if $\bar{s}_k$ is such an inexact Newton step, then the backtracking curve

$$\sigma_k(\eta) \equiv \frac{1 - \eta}{1 - \bar{\eta}_k}\,\bar{s}_k, \qquad \bar{\eta}_k \leq \eta \leq 1, \tag{5.2}$$

satisfies (5.1). The lemma below gives useful conditions under which neither algorithm breaks down in the while-loop once σ_k satisfying (5.1) has been found.

LEMMA 5.1. *At the kth step of Algorithm* MR (*Algorithm* TL), *if* $F(x_k) \neq 0$ *and there exists a* Γ *for which*

$$\|\sigma_k(\eta)\| \leq \Gamma(1-\eta)\|F(x_k)\|, \qquad \bar{\eta}_k \leq \eta \leq 1, \tag{5.3}$$

then the while-loop terminates with

$$1 - \eta_k \geq \min\left\{1 - \bar{\eta}_k, \frac{\theta_{\min}\delta}{\Gamma\|F(x_k)\|}\right\} \tag{5.4}$$

for any $\delta > 0$ *sufficiently small that*

$$\|F(x) - F(x_k) - F'(x_k)(x - x_k)\| \leq \frac{1-t}{\Gamma}\|x - x_k\| \tag{5.5}$$

whenever $x \in N_\delta(x_k)$.

Remark. If σ_k is given by (5.2) and $F(x_k) \neq 0$, then (5.3) holds with $\Gamma \equiv \|\bar{s}_k\|/[(1-\bar{\eta}_k)\|F(x_k)\|]$. It follows from Lemma 5.1 and the preceding discussion that, for either algorithm, if $F(x_k) \neq 0$, then there exists *some* σ_k for which breakdown is avoided at the kth step if and only if there exists *some* inexact Newton step at the level $\bar{\eta}_k$.

Proof. Note that if $\eta \in [\bar{\eta}_k, 1]$ is such that $1-\eta < \delta/(\Gamma\|F(x_k)\|)$, then (5.3), (5.1), and (5.5) imply that $\|\sigma_k(\eta)\| < \delta$ and

$$\begin{aligned}\|F(x_k + \sigma_k(\eta))\| &\leq \|F(x_k) + F'(x_k)\sigma_k(\eta)\| \\ &\qquad + \|F(x_k + \sigma_k(\eta)) - F(x_k) - F'(x_k)\sigma_k(\eta)\| \\ &\leq \eta\|F(x_k)\| + \frac{1-t}{\Gamma}\|\sigma_k(\eta)\|, \\ &\leq [1 - t(1-\eta)]\|F(x_k)\|.\end{aligned}$$

Since $1 - \eta_k$ is reduced by a factor $\theta \leq \theta_{\max} < 1$ at each iteration of the while-loop, it follows that the while-loop terminates.

Suppose that η_k is the final value determined by the while-loop. If $\eta_k = \bar{\eta}_k$, then (5.4) is immediate, so suppose that $\eta_k \neq \bar{\eta}_k$, i.e., the body of the while-loop has been executed at least once. Denoting the penultimate value by η_k^-, we see from the observation above that $1 - \eta_k^- \geq \delta/(\Gamma\|F(x_k)\|)$. Since $1 - \eta_k = \theta(1 - \eta_k^-)$ for some $\theta \geq \theta_{\min}$, it follows that $1 - \eta_k \geq \theta_{\min}\delta/(\Gamma\|F(x_k)\|)$. $\square$

Algorithms MR and TL are clearly special cases of Algorithm GIN, and we use Theorems 3.4 and 3.5 to obtain the global convergence result below.

THEOREM 5.2 (global convergence of algorithms MR and TL). *Assume that Algorithm* MR (*Algorithm* TL) *does not break down. If* x_* *is a limit point of* $\{x_k\}$ *such that there exists a* Γ *independent of k for which*

$$\|\sigma_k(\eta)\| \leq \Gamma(1-\eta)\|F(x_k)\|, \qquad \bar{\eta}_k \leq \eta \leq 1, \tag{5.6}$$

holds whenever x_k *is sufficiently near* x_*, *then* $F(x_*) = 0$ *and* $x_k \to x_*$. *Furthermore,* $\eta_k = \bar{\eta}_k$ *for all sufficiently large* k.

Remark. If $\{x_k\}$ generated by Algorithm MR or Algorithm TL converges to a solution at which F' is invertible and if $\eta_k = \bar{\eta}_k$ for all sufficiently large k, then the ultimate rate of convergence is governed by the choices of the $\bar{\eta}_k$'s as in the local theory of Dembo, Eisenstat, and Steihaug [4]. Analogous statements hold for the results in §§6–8 that are derived from Theorem 5.2.

Proof. It follows immediately from Theorem 3.5 that $x_k \to x_*$.

If $F(x_k) = 0$ for some k, then $F(x_j) = 0$ for all $j \geq k$ and the remainder of the theorem follows easily. Suppose that $F(x_k) \neq 0$ for every k. Let $\delta > 0$ be sufficiently small that (5.6) holds whenever $x_k \in N_\delta(x_*)$ and also

$$\|F(y) - F(x) - F'(x)\,(y - x)\| \leq \frac{1-t}{\Gamma}\,\|y - x\|$$

whenever x, $y \in N_{2\delta}(x_*)$. It follows from Lemma 5.1 that (5.4) holds on termination of the while-loop whenever $x_k \in N_\delta(x_*)$. Set $K \equiv \sup_{x \in N_\delta(x_*)} \|F(x)\|$.

To apply Theorem 3.4, we show that $\sum_{k \geq 0}(1 - \eta_k)$ is divergent. In the case of Algorithm MR, it follows from (5.4) that whenever $x_k \in N_\delta(x_*)$, we have

$$1 - \eta_k \geq \min\left\{1 - \eta_{\max}, \frac{\theta_{\min}\delta}{\Gamma K}\right\}$$

on termination of the while-loop. Since $x_k \to x_*$, there are infinitely many k for which $x_k \in N_\delta(x_*)$; therefore, $\sum_{k \geq 0}(1 - \eta_k)$ is divergent in this case. In the case of Algorithm TL, we set $\epsilon \equiv (1 - u)/\Gamma$ and take δ smaller if necessary so that

$$\|F(y) - F(x) - F'(x)\,(y - x)\| \leq \epsilon\|y - x\| \tag{5.7}$$

whenever x, $y \in N_\delta(x_*)$. Suppose that k_0 is sufficiently large that $x_k \in N_\delta(x_*)$ whenever $k \geq k_0$. Then for $k \geq k_0$, (5.7) and (5.6) give

$$\begin{aligned}
\|F(x_{k+1})\| &\leq \|F(x_k) + F'(x_k)\,\sigma_k(\eta_k)\| \\
&\qquad + \|F(x_{k+1}) - F(x_k) - F'(x_k)\,\sigma_k(\eta_k)\| \\
&\leq \eta_k\|F(x_k)\| + \epsilon\|\sigma_k(\eta_k)\| \\
&\leq [1 - u(1 - \eta_k)]\|F(x_k)\|,
\end{aligned}$$

which implies $\bar{\eta}_{k+1} \leq \eta_k$. Then (5.4) and an easy induction give

$$\begin{aligned}
1 - \eta_{k+1} &\geq \min\left\{1 - \bar{\eta}_{k+1}, \frac{\theta_{\min}\delta}{\Gamma K}\right\} \\
&\geq \min\left\{1 - \eta_k, \frac{\theta_{\min}\delta}{\Gamma K}\right\} \\
&\geq \min\left\{1 - \eta_{k_0}, \frac{\theta_{\min}\delta}{\Gamma K}\right\}.
\end{aligned}$$

It follows that $\sum_{k \geq 0}(1 - \eta_k)$ is divergent in this case as well.

Since $\sum_{k \geq 0}(1 - \eta_k)$ is divergent, Theorem 3.4 implies that $F(x_*) = 0$. Then $F(x_k) \to 0$, and it follows from (5.4) that $\eta_k = \bar{\eta}_k$ whenever k is sufficiently large. □

6. Backtracking methods. In this section, we outline backtracking algorithms that can be easily implemented as practical algorithms once a means of determining initial inexact Newton steps has been provided. These algorithms are based on Algorithm MR in §5; Algorithm TL does not provide a useful paradigm, as noted below. Developments in this section are not used in later sections.

The principal algorithm in this section is the following.

ALGORITHM INB (inexact Newton backtracking method).

LET x_0, $\eta_{\max} \in [0,1)$, $t \in (0,1)$, AND $0 < \theta_{\min} < \theta_{\max} < 1$ BE GIVEN.
FOR $k = 0$ STEP 1 UNTIL ∞ DO:
  CHOOSE $\bar{\eta}_k \in [0, \eta_{\max}]$ AND DETERMINE $\bar{s}_k$ SUCH THAT
    $\|F(x_k) + F'(x_k)\,\bar{s}_k\| \le \bar{\eta}_k \|F(x_k)\|$.
  SET $s_k = \bar{s}_k$ AND $\eta_k = \bar{\eta}_k$.
  WHILE $\|F(x_k + s_k)\| > [1 - t(1-\eta_k)]\,\|F(x_k)\|$ DO:
    CHOOSE $\theta \in [\theta_{\min}, \theta_{\max}]$.
    UPDATE $s_k \leftarrow \theta s_k$ AND $\eta_k \leftarrow 1 - \theta(1-\eta_k)$.
  SET $x_{k+1} = x_k + s_k$.

Algorithm INB is a special case of Algorithm MR. Indeed, for each k, if σ_k is the backtracking curve given previously by (5.2), i.e.,

$$\sigma_k(\eta) \equiv \frac{1-\eta}{1-\bar{\eta}_k}\bar{s}_k, \qquad \bar{\eta}_k \le \eta \le 1, \tag{6.1}$$

then σ_k satisfies (5.1) and an easy induction shows that $s_k = \sigma_k(\eta_k)$ in the while-loop of Algorithm INB. Algorithm MR provides a useful paradigm here because if σ_k is given by (6.1) with $\bar{\eta}_k \in [0, \eta_{\max}]$, then we can show that (5.3) holds for Γ independent of k near a point at which F' is invertible and we can apply Theorem 5.2. Algorithm TL does not provide a useful paradigm because no inequality of the form (5.3) can hold for *all* backtracking curves if $\bar{\eta}_k$ is not bounded away from one.

Algorithm INB does not break down at the kth step if a suitable $\bar{s}_k$ can be found and if either $F(x_k) \ne 0$ or $F'(x_k)$ is invertible. Indeed, suppose we have found $\bar{s}_k$. If $F(x_k) \ne 0$, then it follows from Lemma 5.1 and the subsequent remark that the algorithm does not break down in the while-loop; if $F(x_k) = 0$ and $F'(x_k)$ is invertible, then $\bar{s}_k = 0$ and the while-loop terminates immediately. Note that if $F'(x_k)$ is invertible, then a suitable $\bar{s}_k$ exists and the algorithm does not break down at the kth step. Similar remarks hold for the other algorithms outlined in this section.

THEOREM 6.1 (global convergence of algorithm INB). *Assume that Algorithm* INB *does not break down. If x_* is a limit point of $\{x_k\}$ such that $F'(x_*)$ is invertible, then $F(x_*) = 0$ and $x_k \to x_*$. Furthermore, $s_k = \bar{s}_k$ and $\eta_k = \bar{\eta}_k$ for all sufficiently large k.*

Proof. Set $K \equiv \|F'(x_*)^{-1}\|$ and let $\delta > 0$ be sufficiently small that $F'(x)^{-1}$ exists and $\|F'(x)^{-1}\| \le 2K$ whenever $x \in N_\delta(x_*)$. Suppose that $x_k \in N_\delta(x_*)$, and set $r \equiv F(x_k) + F'(x_k)\,\bar{s}_k$. With $\sigma_k(\eta)$ given by (6.1), we have

$$\begin{aligned} \|\sigma_k(\eta)\| &\le \|F'(x_k)^{-1}\|\,\|F'(x_k)\sigma_k(\eta)\| \\ &\le 2K\frac{1-\eta}{1-\bar{\eta}_k}\|-F(x_k)+r\| \\ &\le 2K\frac{1+\bar{\eta}_k}{1-\bar{\eta}_k}(1-\eta)\|F(x_k)\| \end{aligned}$$

$$\leq 2K\frac{1+\eta_{\max}}{1-\eta_{\max}}(1-\eta)\|F(x_k)\|.$$

Thus (5.3) holds with $\Gamma \equiv 2K(1+\eta_{\max})/(1-\eta_{\max})$, and the theorem follows from Theorem 5.2. □

Application. Backtracking for (exact) Newton's method. When the initial trial step is the Newton step at each iteration, Algorithm INB has a particularly simple form. We take $\eta_{\max} = 0$, so that $\bar{\eta}_k = 0$ and $F'(x_k)\bar{s}_k = -F(x_k)$ for each k. Since $s_k = \sigma_k(\eta_k)$ for σ_k given by (6.1), we have $F'(x_k)s_k = -(1-\eta_k)F(x_k)$, whence

$$\|F(x_k) + F'(x_k)s_k\| = \eta_k\|F(x_k)\|.$$

Thus $pred_k(s_k) = (1-\eta_k)\|F(x_k)\|$, and the while-loop terminates when $ared_k(s_k) \geq t \cdot pred_k(s_k)$ (see §2). Algorithm INB can now be rephrased as follows:

ALGORITHM ENB (exact Newton method with backtracking).
- LET x_0, $t \in (0,1)$, AND $0 < \theta_{\min} < \theta_{\max} < 1$ BE GIVEN.
- FOR $k = 0$ STEP 1 UNTIL ∞ DO:
  - SOLVE $F'(x_k)\, s_k = -F(x_k)$.
  - WHILE $ared_k(s_k) < t \cdot pred_k(s_k)$ DO:
    - CHOOSE $\theta \in [\theta_{\min}, \theta_{\max}]$.
    - UPDATE $s_k \leftarrow \theta s_k$.
  - SET $x_{k+1} = x_k + s_k$.

COROLLARY 6.2 (global convergence of algorithm ENB). *Assume that Algorithm ENB does not break down. If x_* is a limit point of $\{x_k\}$ such that $F'(x_*)$ is invertible, then $F(x_*) = 0$, $x_k \to x_*$, and $s_k = -F'(x_k)^{-1}F(x_k)$, the full Newton step, for all sufficiently large k.*

Extension. Piecewise linear backtracking through inexact Newton steps. Suppose that at the kth iteration of Algorithm MR we have choosen $\bar{\eta}_k$ and have determined inexact Newton steps $s_k^{(1)}, \ldots, s_k^{(m_k)}$ such that

$$\|F(x_k) + F'(x_k)\, s_k^{(j)}\| \leq \eta_k^{(j)}\|F(x_k)\|, \qquad j = 1, \ldots, m_k,$$

where $\eta_k^{(1)}, \ldots, \eta_k^{(m_k)}$ satisfy

$$\eta_{\max} \geq \eta_k^{(1)} > \cdots > \eta_k^{(m_k)} = \bar{\eta}_k.$$

We set $\eta_k^{(0)} \equiv 1$ and $s_k^{(0)} \equiv 0$ and define a piecewise linear curve σ_k by connecting the steps $s_k^{(j)}$:

$$\sigma_k(\eta) \equiv \frac{\eta - \eta_k^{(j)}}{\eta_k^{(j-1)} - \eta_k^{(j)}}\, s_k^{(j-1)} + \frac{\eta_k^{(j-1)} - \eta}{\eta_k^{(j-1)} - \eta_k^{(j)}}\, s_k^{(j)} \quad \text{if } \eta \in [\eta_k^{(j)}, \eta_k^{(j-1)}]. \tag{6.2}$$

We have (5.1) by convexity, and so this σ_k is admissible in Algorithm MR.

THEOREM 6.3. *Assume that Algorithm MR with each σ_k defined by (6.2) does not break down. If x_* is a limit point of $\{x_k\}$ such that $F'(x_*)$ is invertible, then $F(x_*) = 0$ and $x_k \to x_*$. Furthermore, $\eta_k = \bar{\eta}_k$ for all sufficiently large k.*

Proof. We claim that if $F'(x_k)$ is invertible, then

$$\|\sigma_k(\eta)\| \leq \|F'(x_k)^{-1}\| \left(\frac{1+\eta_{\max}}{1-\eta_{\max}}\right)(1-\eta)\|F(x_k)\|, \qquad \bar{\eta}_k \leq \eta \leq 1. \tag{6.3}$$

Indeed, if $\bar{\eta}_k \le \eta \le \eta_k^{(1)}$ ($\le \eta_{\max}$), then

$$\begin{aligned}\|F'(x_k)\,\sigma_k(\eta)\| &\le \|F(x_k)\| + \|F(x_k) + F'(x_k)\,\sigma_k(\eta)\| \\ &\le (1+\eta)\|F(x_k)\| \\ &\le \frac{1+\eta_{\max}}{1-\eta_{\max}}\,(1-\eta)\|F(x_k)\|;\end{aligned}$$

and if $\eta_k^{(1)} \le \eta \le \eta_k^{(0)}$ $(=1)$, then

$$\begin{aligned}\|F'(x_k)\,\sigma_k(\eta)\| &= \left\|F'(x_k)\,\left[\frac{1-\eta}{1-\eta_k^{(1)}}\,s_k^{(1)}\right]\right\| \\ &\le \frac{1-\eta}{1-\eta_k^{(1)}}\,\{\|F(x_k)\| + \|F(x_k) + F'(x_k)\,s_k^{(1)}\|\} \\ &\le \frac{1-\eta}{1-\eta_k^{(1)}}\,(1+\eta_k^{(1)})\|F(x_k)\| \\ &\le \frac{1+\eta_{\max}}{1-\eta_{\max}}\,(1-\eta)\|F(x_k)\|.\end{aligned}$$

Since $\|\sigma_k(\eta)\| \le \|F'(x_k)^{-1}\|\,\|F'(x_k)\,\sigma_k(\eta)\|$, inequality (6.3) follows.

It is clear from (6.3) that an inequality of the form (5.3) holds for x_k sufficiently near a point x_* such that $F'(x_*)$ is invertible, and the theorem follows from Theorem 5.2. □

7. Equality curve methods. In this section, we consider instances of Algorithms MR and TL in which each curve σ_k is continuous, satisfies $\sigma_k(1) = 0$, and is such that (5.1) holds with equality, i.e.,

$$\|F(x_k) + F'(x_k)\,\sigma_k(\eta)\| = \eta\|F(x_k)\|, \qquad \bar{\eta}_k \le \eta \le 1. \tag{7.1}$$

In §8 we give important applications in which such σ_k's are easily determined. Here, we reformulate Algorithms MR and TL appropriately for such σ_k's and establish a somewhat specialized global convergence analysis, assuming that the norm is an inner product norm. These reformulated algorithms can be readily implemented as practical algorithms once a means of determining such σ_k's has been specified.

At the kth iteration of Algorithms MR and TL, the while-loop terminates if

$$\|F(x_k + \sigma_k(\eta_k))\| \le [1 - t(1-\eta_k)]\,\|F(x_k)\|. \tag{7.2}$$

By analogy with (2.4), we define

$$\begin{aligned}ared_k(\eta_k) &\equiv \|F(x_k)\| - \|F(x_k + \sigma_k(\eta_k))\|, \\ pred_k(\eta_k) &\equiv \|F(x_k)\| - \|F(x_k) + F'(x_k)\sigma_k(\eta_k)\|.\end{aligned}$$

If (7.1) holds, then $pred_k(\eta_k) = (1-\eta_k)\|F(x_k)\|$, and (7.2) is equivalent to

$$ared_k(\eta_k) \ge t \cdot pred_k(\eta_k).$$

This leads to the following reformulations of Algorithms MR and TL.

ALGORITHM ECMR (equality curve minimum reduction method).

LET x_0, $\eta_{\max} \in [0,1)$, $t \in (0,1)$, AND $0 < \theta_{\min} < \theta_{\max} < 1$ BE GIVEN.
FOR $k = 0$ STEP 1 UNTIL ∞ DO:
  CHOOSE $\bar{\eta}_k \in [0, \eta_{\max}]$.
  DETERMINE A CONTINUOUS σ_k SUCH THAT (7.1) HOLDS
    AND $\sigma_k(1) = 0$; SET $\eta_k = \bar{\eta}_k$.
  WHILE $ared_k(\eta_k) < t \cdot pred_k(\eta_k)$ DO:
    CHOOSE $\theta \in [\theta_{\min}, \theta_{\max}]$.
    UPDATE $\eta_k \leftarrow 1 - \theta(1 - \eta_k)$.
  SET $x_{k+1} = x_k + \sigma_k(\eta_k)$.

ALGORITHM ECTL (equality curve trust level method).

LET x_0, $\bar{\eta}_0 \in [0,1)$, $0 < t \le u < 1$, AND $0 < \theta_{\min} < \theta_{\max} < 1$ BE GIVEN.
FOR $k = 0$ STEP 1 UNTIL ∞ DO:
  DETERMINE A CONTINUOUS σ_k SUCH THAT (7.1) HOLDS
    AND $\sigma_k(1) = 0$; SET $\eta_k = \bar{\eta}_k$.
  WHILE $ared_k(\eta_k) < t \cdot pred_k(\eta_k)$ DO:
    CHOOSE $\theta \in [\theta_{\min}, \theta_{\max}]$.
    UPDATE $\eta_k \leftarrow 1 - \theta(1 - \eta_k)$.
  SET $x_{k+1} = x_k + \sigma_k(\eta_k)$.
  IF $ared_k(\eta_k) \ge u \cdot pred_k(\eta_k)$ CHOOSE $\bar{\eta}_{k+1} \in [0, \eta_k]$;
   ELSE CHOOSE $\bar{\eta}_{k+1} \in [0, 1 - \theta_{\min}(1 - \eta_k)]$.

Theorem 5.2 provides a general global convergence result for Algorithms ECMR and ECTL, but we develop a more specialized result under the assumption that the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot \rangle$, i.e., $\|v\| = \langle v, v \rangle^{1/2}$ for $v \in \mathbf{R}^n$.

Define $f(x) \equiv \frac{1}{2}\|F(x)\|^2$ for $x \in \mathbf{R}^n$ and note that the Fréchet derivative f' of f at x is given by $f'(x) = \langle F'(x)^* F(x), \cdot \rangle$, where $F'(x)^*$ is the adjoint operator of $F'(x)$ with respect to $\langle \cdot, \cdot \rangle$. Thus $F'(x)^* F(x)$ is the *gradient* of f at x with respect to $\langle \cdot, \cdot \rangle$, and $-F'(x)^* F(x)$ is the *steepest descent direction* (see Kantorovich and Akilov [11, pp. 462–463]).

LEMMA 7.1. *If x and s are such that $\|F(x) + F'(x)s\| < \|F(x)\|$, then*

$$\langle F'(x)^* F(x), s \rangle < 0,$$

i.e., s is a descent direction for f at x.

Proof. Writing $r \equiv F(x) + F'(x)s$, we have

$$\langle F'(x)^* F(x), s \rangle = \langle F(x), -F(x) + r \rangle \le -\|F(x)\|^2 + \|F(x)\|\|r\| < 0. \qquad \square$$

LEMMA 7.2. *At the kth step of Algorithm* ECMR (*Algorithm* ECTL), *assume that $F(x_k) \neq 0$ and set*

$$\gamma_k \equiv \inf_{\bar{\eta}_k \le \eta < 1} \frac{|\langle F(x_k), F'(x_k)\sigma_k(\eta) \rangle|}{\|F(x_k)\|\|F'(x_k)\sigma_k(\eta)\|}. \tag{7.3}$$

If $\gamma_k > 0$ and $F'(x_k)$ is invertible, then

$$\|\sigma_k(\eta)\| \le \frac{2\|F'(x_k)^{-1}\|}{\gamma_k}(1 - \eta)\|F(x_k)\| \tag{7.4}$$

for $\max\{\bar{\eta}_k, (1 - \gamma_k^2)^{1/2}\} \le \eta \le 1$.

Proof. For $\bar{\eta}_k \le \eta \le 1$, we have

$$\begin{aligned} \eta^2\|F(x_k)\|^2 &= \|F(x_k) + F'(x_k)\,\sigma_k(\eta)\|^2 \\ &= \|F(x_k)\|^2 + 2\langle F(x_k), F'(x_k)\,\sigma_k(\eta)\rangle + \|F'(x_k)\,\sigma_k(\eta)\|^2. \end{aligned} \tag{7.5}$$

Define

$$\gamma(\eta) \equiv \frac{|\langle F(x_k), F'(x_k)\,\sigma_k(\eta)\rangle|}{\|F(x_k)\|\|F'(x_k)\,\sigma_k(\eta)\|}$$

and note that $\langle F(x_k), F'(x_k)\,\sigma_k(\eta)\rangle < 0$ by Lemma 7.1. Then (7.5) gives

$$\|F'(x_k)\,\sigma_k(\eta)\|^2 - 2\gamma(\eta)\|F(x_k)\|\|F'(x_k)\,\sigma_k(\eta)\| + (1-\eta^2)\|F(x_k)\|^2 = 0.$$

This is a quadratic equation in $\|F'(x_k)\,\sigma_k(\eta)\|$, the solutions of which are

$$\gamma(\eta)\|F(x_k)\|\left\{1 \pm \sqrt{1 - \frac{1-\eta^2}{\gamma(\eta)^2}}\right\}.$$

Since $\sigma_k(\eta)$ is continuous and $\sigma_k(1) = 0$, we must have

$$\|F'(x_k)\,\sigma_k(\eta)\| = \gamma(\eta)\|F(x_k)\|\left\{1 - \sqrt{1 - \frac{1-\eta^2}{\gamma(\eta)^2}}\right\} \tag{7.6}$$

whenever η is sufficiently near one. In particular, since $(1-\eta^2)/\gamma(\eta)^2 < 1$ for $\max\{\bar{\eta}_k, (1-\gamma_k^2)^{1/2}\} < \eta < 1$, we must have (7.6) for $\max\{\bar{\eta}_k, (1-\gamma_k^2)^{1/2}\} \le \eta < 1$. Since $1 - \sqrt{1-\epsilon} \le \epsilon$ whenever $0 \le \epsilon \le 1$, it follows that

$$\|F'(x_k)\,\sigma_k(\eta)\| \le \gamma(\eta)\|F(x_k)\|\frac{1-\eta^2}{\gamma(\eta)^2} \le \frac{2}{\gamma(\eta)}(1-\eta)\|F(x_k)\|$$

for $\max\{\bar{\eta}_k, (1-\gamma_k^2)^{1/2}\} \le \eta < 1$. Since $\|\sigma_k(\eta)\| \le \|F'(x_k)^{-1}\|\,\|F'(x_k)\,\sigma_k(\eta)\|$, inequality (7.4) follows. □

It is clear from (7.3) that $\gamma_k > 0$ if and only if the vectors $F'(x_k)\,\sigma_k(\eta)$ are bounded away from orthogonality with $F(x_k)$. Moreover, if $F'(x_k)$ is invertible, then $\gamma_k > 0$ if and only if the vectors $\sigma_k(\eta)$, which are descent directions for f at x_k, are bounded away from orthogonality with the steepest descent direction $-F'(x_k)^*\,F(x_k)$. This follows from

$$\begin{aligned} \kappa(F'(x_k))^{-1}\,\frac{|\langle F'(x_k)^*\,F(x_k), \sigma_k(\eta)\rangle|}{\|F'(x_k)^*\,F(x_k)\|\|\sigma_k(\eta)\|} &\le \frac{|\langle F(x_k), F'(x_k)\,\sigma_k(\eta)\rangle|}{\|F(x_k)\|\|F'(x_k)\,\sigma_k(\eta)\|} \\ &\le \kappa(F'(x_k))\,\frac{|\langle F'(x_k)^*\,F(x_k), \sigma_k(\eta)\rangle|}{\|F'(x_k)^*\,F(x_k)\|\|\sigma_k(\eta)\|}, \end{aligned}$$

where $\kappa(F'(x_k)) \equiv \|F'(x_k)\|\|F'(x_k)^{-1}\|$ is the condition number of $F'(x_k)$.

If $F(x_k) \ne 0$, $F'(x_k)$ is invertible, and σ_k is such that $\gamma_k > 0$, then it follows from Lemmas 7.2 and 5.1 that neither Algorithm ECMR nor Algorithm ECTL breaks down in the while-loop. In fact, if $F'(x_k)$ is invertible, then it is always possible to avoid breakdown in either algorithm. Indeed, one can choose $\sigma_k(\eta) \equiv (1-\eta)[-F'(x_k)^{-1}F(x_k)]$, for which (7.1) holds. For this σ_k, if $F(x_k) \ne 0$, then

$\gamma_k = 1$, and if $F(x_k) = 0$, then $\sigma_k(\eta) \equiv 0$. In either case, neither algorithm breaks down in the while-loop.

The specialized global convergence result below is an immediate corollary of Theorem 5.2 and Lemma 7.2.

THEOREM 7.3 (global convergence of algorithms ECMR and ECTL). *Assume that Algorithm* ECMR (*Algorithm* ECTL) *does not break down. If x_* is a limit point of $\{x_k\}$ such that $F'(x_*)$ is invertible and if there exists a $\gamma > 0$ independent of k such that*

$$\gamma_k \equiv \inf_{\bar{\eta}_k \le \eta < 1} \frac{|\langle F(x_k), F'(x_k)\,\sigma_k(\eta)\rangle|}{\|F(x_k)\|\|F'(x_k)\,\sigma_k(\eta)\|} \ge \gamma \tag{7.7}$$

whenever $F(x_k) \ne 0$ and x_k is sufficiently near x_*, then $F(x_*) = 0$ and $x_k \to x_*$. Furthermore, $\eta_k = \bar{\eta}_k$ for all sufficiently large k.

Proof. Set $K \equiv \|F'(x_*)^{-1}\|$ and assume that $\delta > 0$ is sufficiently small that $F'(x)^{-1}$ exists and $\|F'(x)^{-1}\| \le 2K$ whenever $x \in N_\delta(x_*)$ and, furthermore, that (7.7) holds whenever $F(x_k) \ne 0$ and $x_k \in N_\delta(x_*)$.

Suppose that $x_k \in N_\delta(x_*)$. If $F(x_k) \ne 0$, then (7.7) and Lemma 7.2 give

$$\|\sigma_k(\eta)\| \le \frac{4K}{\gamma}(1-\eta)\|F(x_k)\| \tag{7.8}$$

for $\max\{\bar{\eta}_k, (1-\gamma^2)^{1/2}\} \le \eta \le 1$. If $F(x_k) = 0$, then $\sigma_k(\eta) \equiv 0$ since $F'(x_k)$ is invertible; thus equation (7.8) still holds. It follows easily that there is a Γ for which (5.3) holds whenever $x_k \in N_\delta(x_*)$, and the theorem follows from Theorem 5.2. □

8. Applications of equality curve methods. In this section, we give applications of the results in §7, focusing on ways of determining the curves σ_k in Algorithms ECMR and ECTL. In the first application, our main purpose is to outline ways of determining these curves in certain Newton iterative methods; we also show how to obtain a dogleg globalization of (exact) Newton's method using Algorithm ECTL. In the second application, we show how Levenberg–Marquardt globalizations of Newton's method can be obtained from Algorithms ECMR and ECTL.

We assume throughout that $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$.

Application 1. Piecewise linear backtracking through residual minimizing steps. Suppose that the Newton equation (1.2) is solved approximately using a *residual minimizing* iterative method, i.e., a method for solving a linear system $Au = b$ that begins with an initial approximate solution u_0 (and residual $r_0 \equiv b - Au_0$) and, at the ith iteration, determines a subspace $\mathcal{K}_i \subseteq \mathbf{R}^n$ and a correction

$$z_i \in \arg\min_{z\in\mathcal{K}_i} \|b - A(u_0 + z)\|.$$

There are many such methods. Two familiar examples, in which $\|\cdot\| = \|\cdot\|_2$, are the GMRES method of Saad and Schultz [15], with $\mathcal{K}_i \equiv \mathrm{span}\{r_0, Ar_0, \ldots, A^{i-1}r_0\}$, and the conjugate gradient method applied to the normal equations $A^TAx = A^Tb$, or CGNR (see Elman [9]), with $\mathcal{K}_i \equiv \mathrm{span}\{A^Tr_0, (A^TA)A^Tr_0, \ldots, (A^TA)^{i-1}A^Tr_0\}$.

Here, we require only that $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots$. Then $\rho_i \equiv \|b - A(u_0 + z_i)\|$ is nonincreasing in i. The following lemma shows that if $\rho_i > \rho_j$ for some $i < j$, then the residual norm is strictly monotone decreasing along the line segment from $u_0 + z_i$ to $u_0 + z_j$ and gives a parametrization of this line segment in terms of the residual norm.

LEMMA 8.1. *Suppose that $\rho_i > \rho_j$. For $\rho_i \geq \rho \geq \rho_j$, set*

$$\tau(\rho) \equiv \left(\frac{\rho^2 - \rho_j^2}{\rho_i^2 - \rho_j^2}\right)^{\frac{1}{2}} \quad \textit{and} \quad z(\rho) \equiv \tau(\rho) z_i + (1 - \tau(\rho)) z_j.$$

Then $\|b - A(u_0 + z(\rho))\| = \rho$ for $\rho_i \geq \rho \geq \rho_j$.

Proof. Defining $\|w\|_A \equiv \|Aw\|$ for $w \in \mathbf{R}^n$, we have

$$\|b - A(u_0 + z)\| = \|u - u_0 - z\|_A.$$

Therefore, for any l, we have $z_l = P_l(u - u_0)$, where P_l denotes $\|\cdot\|_A$-orthogonal projection onto $\mathcal{K}_l$, and

$$\rho_l = \|u - u_0 - z_l\|_A = \|P_l^{\perp}(u - u_0)\|_A,$$

where $P_l^{\perp} \equiv I - P_l$. Also, since $i < j$, we have $\mathcal{K}_i \subset \mathcal{K}_j$ and $P_j^{\perp} P_i^{\perp} = P_i^{\perp} P_j^{\perp} = P_j^{\perp}$. It follows that for $\rho_i \geq \rho \geq \rho_j$,

$$\begin{aligned}
\|b - A(u_0 + z(\rho))\|^2 &= \|u - u_0 - z(\rho)\|_A^2 \\
&= \|\tau(\rho) P_i^{\perp}(u - u_0) + (1 - \tau(\rho)) P_j^{\perp}(u - u_0)\|_A^2 \\
&= \tau(\rho)^2 \|P_i^{\perp}(u - u_0)\|_A^2 + (1 - \tau(\rho)^2) \|P_j^{\perp}(u - u_0)\|_A^2 \\
&= \tau(\rho)^2 \rho_i^2 + (1 - \tau(\rho)^2) \rho_j^2 \\
&= \rho^2. \qquad \square
\end{aligned}$$

We determine a curve σ_k at the kth step of Algorithm ECMR or Algorithm ECTL as follows: If $F(x_k) = 0$, then $\sigma_k(\eta) \equiv 0$ for $\bar{\eta}_k \leq \eta \leq 1$. If $F(x_k) \neq 0$, then we apply a residual minimizing method to the Newton equation (1.2), beginning with zero as the initial approximate solution, to obtain approximate solutions $s_k^{(0)} = 0$, $s_k^{(1)}, \ldots, s_k^{(m_k)}$, which are not necessarily *consecutive* iterates but such that

$$1 = \eta_k^{(0)} > \eta_k^{(1)} > \cdots > \eta_k^{(m_k - 1)} > \bar{\eta}_k \geq \eta_k^{(m_k)},$$

where

$$\eta_k^{(i)} \equiv \frac{\|F(x_k) + F'(x_k)\, s_k^{(i)}\|}{\|F(x_k)\|}, \qquad i = 0, \ldots, m_k.$$

If $\bar{\eta}_k > \eta_k^{(m_k)}$, then we force $\bar{\eta}_k = \eta_k^{(m_k)}$ either by redefining $\bar{\eta}_k$ to be $\eta_k^{(m_k)}$ or by replacing $s_k^{(m_k)}$ with an appropriate convex combination of $s_k^{(m_k)}$ and $s_k^{(m_k-1)}$, which is easily done using Lemma 8.1. We take σ_k to be the piecewise linear curve connecting $s_k^{(0)}, \ldots, s_k^{(m_k)}$. It follows from Lemma 8.1 that this curve can easily be parametrized continuously in η such that (7.1) holds and $\sigma_k(1) = 0$.

We give global convergence results for Algorithms ECMR and ECTL with each σ_k determined in this way.

LEMMA 8.2. *If $\|F(x_k)\| \neq 0$ and $\|F(x_k) + F'(x_k)\, s\| \leq \eta \|F(x_k)\|$ for some s and $\eta \in [0, 1)$, then*

$$\frac{|\langle F(x_k), F'(x_k)\, s\rangle|}{\|F(x_k)\| \|F'(x_k)\, s\|} \geq \frac{1 - \eta}{1 + \eta}.$$

Proof. Writing $r \equiv F(x_k) + F'(x_k)\, s$, we have $\|r\| \le \eta \|F(x_k)\| < \|F(x_k)\|$. Then $F'(x_k)\, s \ne 0$, and

$$\frac{|\langle F(x_k), F'(x_k)\, s\rangle|}{\|F(x_k)\|\|F'(x_k)\, s\|} = \frac{|\langle F(x_k), -F(x_k) + r\rangle|}{\|F(x_k)\|\| - F(x_k) + r\|}$$
$$\ge \frac{\|F(x_k)\|^2 - \|F(x_k)\|\|r\|}{\|F(x_k)\|\left(\|F(x_k)\| + \|r\|\right)} \ge \frac{1-\eta}{1+\eta}. \qquad \square$$

THEOREM 8.3. *Assume that Algorithm* ECMR *with each σ_k determined as above does not break down and that $\eta_k^{(1)} \le \eta_{\max}$ whenever $F(x_k) \ne 0$. If x_* is a limit point of $\{x_k\}$ such that $F'(x_*)$ is invertible, then $F(x_*) = 0$ and $x_k \to x_*$. Furthermore, $\eta_k = \bar{\eta}_k$ for all sufficiently large k.*

Proof. We claim that if $F(x_k) \ne 0$ and $\eta_k^{(1)} \le \eta_{\max}$, then

$$\gamma_k \equiv \inf_{\bar{\eta}_k \le \eta < 1} \frac{|\langle F(x_k), F'(x_k)\, \sigma_k(\eta)\rangle|}{\|F(x_k)\|\|F'(x_k)\, \sigma_k(\eta)\|} \ge \gamma \equiv \frac{1-\eta_{\max}}{1+\eta_{\max}}.$$

Indeed, if $\eta_k^{(1)} \le \eta < 1$, then $\sigma_k(\eta) = \tau s_k^{(1)}$ for some $\tau \in (0, 1]$, and Lemma 8.2 gives

$$\frac{|\langle F(x_k), F'(x_k)\, \sigma_k(\eta)\rangle|}{\|F(x_k)\|\|F'(x_k)\, \sigma_k(\eta)\|} = \frac{|\langle F(x_k), F'(x_k)\, s_k^{(1)}\rangle|}{\|F(x_k)\|\|F'(x_k)\, s_k^{(1)}\|} \ge \frac{1-\eta_k^{(1)}}{1+\eta_k^{(1)}} \ge \frac{1-\eta_{\max}}{1+\eta_{\max}}.$$

If $\bar{\eta}_k \le \eta \le \eta_k^{(1)}$, then it follows from Lemma 8.2 that

$$\frac{|\langle F(x_k), F'(x_k)\, \sigma_k(\eta)\rangle|}{\|F(x_k)\|\|F'(x_k)\, \sigma_k(\eta)\|} \ge \frac{1-\eta}{1+\eta} \ge \frac{1-\eta_k^{(1)}}{1+\eta_k^{(1)}} \ge \frac{1-\eta_{\max}}{1+\eta_{\max}}.$$

The theorem now follows from Theorem 7.3. $\square$

To treat Algorithm ECTL, we formulate the following condition on F and the residual minimizing method (with zero as the initial approximate solution) near a point x_*.

Nonorthogonality condition. There exist $\delta > 0$ and $\lambda > 0$ such that if $x \in N_\delta(x_*)$ and $F(x) \ne 0$, then $F'(x)\, z \ne 0$ for *some* $z \in \mathcal{K}_1$ and

$$\max_{\substack{z \in \mathcal{K}_1 \\ F'(x)\, z = 0}} \frac{|\langle F(x), F'(x)\, z\rangle|}{\|F(x)\|\|F'(x)\, z\|} \ge \lambda.$$

Remark. This is a condition that $F'(x)(\mathcal{K}_1)$ be uniformly bounded away from orthogonality with $F(x)$ for x near x_*. As in the remark after Lemma 7.2, if $F'(x)$ is invertible near x_*, then it is equivalent to a condition that $\mathcal{K}_1$ be uniformly bounded away from orthogonality with $-F'(x)^* F(x)$, the steepest descent direction for $f(x) \equiv \frac{1}{2}\|F(x)\|^2$, for x near x_*. Since F' is continuous, this condition holds, e.g., if the method is CGNR and $F'(x_*)$ is invertible, or if the method is GMRES and $F'(x_*) + F'(x_*)^T$ is positive definite.

THEOREM 8.4. *Assume that Algorithm* ECTL *with each σ_k determined as above does not break down. If x_* is a limit point of $\{x_k\}$ such that $F'(x_*)$ is invertible and the nonorthogonality condition holds, then $F(x_*) = 0$ and $x_k \to x_*$. Furthermore, $\eta_k = \bar{\eta}_k$ for all sufficiently large k.*

Proof. Let $\delta > 0$ and $\lambda \in [0,1)$ be as in the nonorthogonality condition, and assume that $x_k \in N_\delta(x_*)$ and $F(x_k) \neq 0$. By the nonorthogonality condition, we have

$$\begin{aligned}
\min_{z\in\mathcal{K}_1} \|F(x_k) + F'(x_k)\,z\| &= \min_{\substack{z\in\mathcal{K}_1 \\ F'(x_k)z=0}} \|F(x_k) + F'(x_k)\,z\| \\
&= \min_{\substack{z\in\mathcal{K}_1 \\ F'(x_k)z=0}} \min_{\mu\in\mathbf{R}} \|F(x_k) + F'(x_k)\,(\mu z)\| \\
&= \min_{\substack{z\in\mathcal{K}_1 \\ F'(x_k)z=0}} \left(\|F(x_k)\|^2 - \frac{\langle F(x_k), F'(x_k)\,z\rangle^2}{\|F'(x_k)\,z\|^2} \right)^{\frac{1}{2}} \\
&= \left(1 - \max_{\substack{z\in\mathcal{K}_1 \\ F'(x_k)z=0}} \frac{\langle F(x_k), F'(x_k)\,z\rangle^2}{\|F(x_k)\|^2 \|F'(x_k)\,z\|^2} \right)^{\frac{1}{2}} \|F(x_k)\| \\
&\le \sqrt{1-\lambda^2}\,\|F(x_k)\|.
\end{aligned}$$

Then we must have $\eta_k^{(1)} \le \sqrt{1-\lambda^2}$ and, by an argument similar to that used in the proof of Theorem 8.3,

$$\gamma_k \equiv \inf_{\bar\eta_k \le \eta < 1} \frac{|\langle F(x_k), F'(x_k)\,\sigma_k(\eta)\rangle|}{\|F(x_k)\| \|F'(x_k)\,\sigma_k(\eta)\|} \ge \gamma \equiv \frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}.$$

The theorem now follows from Theorem 7.3. □

The *dogleg curve* is widely used in trust region globalizations of (exact) Newton's method; see, e.g., Dennis and Schnabel [5, Chap. 6]. At x_k, this curve is the piecewise linear curve connecting zero, the minimizer of $\|F(x_k) + F'(x_k)\,s\|_2$ in the steepest descent direction $-F'(x_k)^T F(x_k)$ (the steepest descent step), and the Newton step $-F'(x_k)^{-1}F(x_k)$. With $\|\cdot\| = \|\cdot\|_2$, we can determine σ_k as a subset of this curve in Algorithm ECTL. Indeed, if we choose $\mathcal{K}_1 \equiv \text{span}\{-F'(x_k)^T F(x_k)\}$ and $\mathcal{K}_2 \equiv \mathbf{R}^n$ and take zero as the initial approximate solution, then the first and second residual minimizing iterates are the steepest descent step and the Newton step, respectively, and we obtain the dogleg curve.[2] Consequently, we can take σ_k to be the part of the dogleg curve from zero to the point at which the local linear model norm is $\bar\eta_k \|F(x_k)\|_2$. Since the nonorthogonality condition holds for this choice of $\mathcal{K}_1$ if $F'(x_*)$ is invertible, the corollary below follows immediately from Theorem 8.4.

COROLLARY 8.5. *Assume that Algorithm* ECTL *with each σ_k obtained from the dogleg curve does not break down. If x_* is a limit point of $\{x_k\}$ such that $F'(x_*)$ is invertible, then $F(x_*) = 0$ and $x_k \to x_*$. Furthermore, $\eta_k = \bar\eta_k$ for all sufficiently large k.*

Application 2. Levenberg–Marquardt globalizations of (exact) Newton's method. The *Levenberg–Marquardt curve* is also often used in trust region globalizations of Newton's method; see, e.g, Dennis and Schnabel [5, Chap. 6]. At x_k, this curve is swept out by arg $\min_{\|s\|_2 \le \delta} \|F(x_k) + F'(x_k)\,s\|_2$, $0 < \delta < \infty$, and is usually given by

$$\sigma_k(\mu) \equiv -\left[F'(x_k)^T F'(x_k) + \mu I\right]^{-1} F'(x_k)^T F(x_k), \qquad 0 \le \mu < \infty. \tag{8.1}$$

[2] The dogleg curve can also be obtained using CGNR: With zero as the initial approximate solution, the first CGNR iteration gives the steepest descent step, and the Newton step is obtained after no more than n iterations.

Note that $\sigma_k(0) = -F'(x_k)^{-1}F(x_k)$, the Newton step, and that $\lim_{\mu\to\infty} \sigma_k(\mu) = 0$. It is well known that

$$\eta(\mu) \equiv \frac{\|F(x_k) + F'(x_k)\,\sigma_k(\mu)\|_2}{\|F(x_k)\|_2}$$

is strictly monotone increasing in μ; furthermore, $\eta(0) = 0$ and $\lim_{\mu\to\infty} \eta(\mu) = 1$. Therefore, we can reparametrize the curve in η and, setting $\sigma_k(1) \equiv 0$, obtain a continuous curve σ_k such that $\sigma_k(1) = 0$ and

$$\|F(x_k) + F'(x_k)\,\sigma_k(\eta)\|_2 = \eta\|F(x_k)\|_2, \qquad 0 \le \eta \le 1.$$

We obtain Levenberg–Marquardt globalizations of Newton's method from Algorithms ECMR and ECTL by taking $\|\cdot\| = \|\cdot\|_2$ and taking each σ_k to be the subcurve of the Levenberg–Marquardt curve from zero to the point at which the local linear model norm is $\bar{\eta}_k\|F(x_k)\|_2$.

THEOREM 8.6. *Assume that Algorithm* ECMR (*Algorithm* ECTL) *with each* σ_k *obtained from the Levenberg–Marquardt curve does not break down. If* x_* *is a limit point of* $\{x_k\}$ *such that* $F'(x_*)$ *is invertible, then* $F(x_*) = 0$ *and* $x_k \to x_*$. *Furthermore,* $\eta_k = \bar{\eta}_k$ *for all sufficiently large* k.

Proof. Set $K \equiv \max\{\|F'(x_*)\|_2, \|F'(x_*)^{-1}\|_2\}$ and let $\delta > 0$ be sufficiently small that $F'(x)^{-1}$ exists and $\max\{\|F'(x)\|_2, \|F'(x)^{-1}\|_2\} \le 2K$ whenever $x \in N_\delta(x_*)$.

Suppose that $x_k \in N_\delta(x_*)$ and $F(x_k) \ne 0$. We derive a lower bound on γ_k in (7.7) with $\bar{\eta}_k = 0$; this will also be a lower bound for any $\bar{\eta}_k \in [0, 1)$. We work with the usual parametrization (8.1). Note that

$$\begin{aligned} F'(x_k)\,\sigma_k(\mu) &= -F'(x_k)\left[F'(x_k)^T F'(x_k) + \mu I\right]^{-1} F'(x_k)^T F(x_k) \\ &= -\left\{I + \mu\left[F'(x_k)F'(x_k)^T\right]^{-1}\right\}^{-1} F(x_k) \\ &= -Q\left(I + \mu\Lambda^{-1}\right)^{-1} Q^T F(x_k), \end{aligned} \tag{8.2}$$

where $F'(x_k)F'(x_k)^T = Q\Lambda Q^T$, Q is orthogonal, and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with each $\lambda_i > 0$. Setting $\lambda_{\max} \equiv \max_i \lambda_i$, $\lambda_{\min} \equiv \min_i \lambda_i$, and $w \equiv (w_1, \dots, w_n)^T \equiv Q^T F(x_k)$, we have from (7.7) and (8.2) that

$$\begin{aligned} \gamma_k &= \inf_{0\le\mu<\infty} \frac{\left|w^T\left(I + \mu\Lambda^{-1}\right)^{-1} w\right|}{\|w\|_2 \|\left(I + \mu\Lambda^{-1}\right)^{-1} w\|_2} \\ &= \inf_{0\le\mu<\infty} \frac{\sum_{i=1}^n \frac{\lambda_i}{\lambda_i+\mu} w_i^2}{\left(\sum_{i=1}^n w_i^2\right)^{1/2}\left[\sum_{i=1}^n \left(\frac{\lambda_i}{\lambda_i+\mu}\right)^2 w_i^2\right]^{1/2}} \\ &\ge \inf_{0\le\mu<\infty} \left(\frac{\lambda_{\min}}{\lambda_{\max}+\mu}\right) \Big/ \left(\frac{\lambda_{\max}}{\lambda_{\min}+\mu}\right) \\ &= \inf_{0\le\mu<\infty} \frac{\lambda_{\min}}{\lambda_{\max}} \frac{\lambda_{\min}+\mu}{\lambda_{\max}+\mu} \\ &\ge \left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^2 \\ &= \left(\frac{1}{\|F'(x_k)^{-1}\|_2 \|F'(x_k)\|_2}\right)^4 \end{aligned}$$

$$\geq \frac{1}{256K^8}.$$

Thus (7.7) holds with $\gamma = (256K^8)^{-1}$ whenever $x_k \in N_\delta(x_*)$ and $F(x_k) \neq 0$, and the theorem follows from Theorem 7.3. □

9. Postview. In this paper, we have outlined inexact Newton methods with strong global convergence properties. Our purposes have been the following:

- to introduce new algorithms that will be useful in implementing Newton iterative methods and other Newton-like methods in which the Newton equation (1.2) is solved inaccurately;
- to provide a general framework for treating globalizations of Newton's method and related methods.

Our most general method is Algorithm GIN in §2; all subsequent algorithms are special cases of it. The analysis for Algorithm GIN in §3 provides the basis for all global convergence results in the sequel; for all methods, except the paradigm methods in §5, these include a basic result to the effect that, under reasonable assumptions, if a sequence of iterates has a limit point at which F' is invertible, then that limit point is a solution of (1.1) and the iterates converge to it. The developments in §§2–3 are first applied directly in §4 to give global convergence results for trust region methods and global approximate Newton methods. They are then used in §5 to develop a theoretical basis for Algorithms MR and TL, which, in turn, serve as paradigms for algorithms in §§6–8 that are of ultimate interest.

The algorithms in §§6–8 test initial inexact Newton steps and, if they are unsatisfactory, determine new steps in specified ways until satisfactory steps are found. For each of these algorithms, we show, in addition to the basic global convergence result above, that initial inexact Newton steps are ultimately satisfactory near a solution at which F' is invertible and, therefore, the convergence can be made as fast as desired, up to the rate of Newton's method, by choosing appropriately small initial inexact Newton levels.

The algorithms in §6 are backtracking methods. The principal method is Algorithm INB for backtracking from initial inexact Newton steps that achieve a specified minimum reduction of the local linear model norm but otherwise are arbitrary. This algorithm is based on Algorithm MR. It can easily be implemented as a practical algorithm once a means of determining initial inexact Newton steps is given. As a special case of Algorithm INB, we give a backtracking globalization of (exact) Newton's method in Algorithm ENB. We also discuss extensions of Algorithm INB to piecewise linear backtracking through inexact Newton steps.

The algorithms in §§7–8 are equality curve methods. The basic algorithms are Algorithms ECMR and ECTL in §7, which are based on Algorithms MR and TL, respectively. To implement these, one must provide a means of specifying continuous curves of inexact Newton steps along which inexact Newton conditions are satisfied with equality. Applications are given in §8 in which such curves are easily and naturally determined. The principal application is to Newton iterative methods in which the Newton equation (1.2) is solved approximately using a residual minimizing iterative method; we also show how to obtain dogleg and Levenberg–Marquardt globalizations of (exact) Newton's method.

We close with Fig. 1 below, which gives an organizational chart of the methods and applications in this paper and their interrelationships.

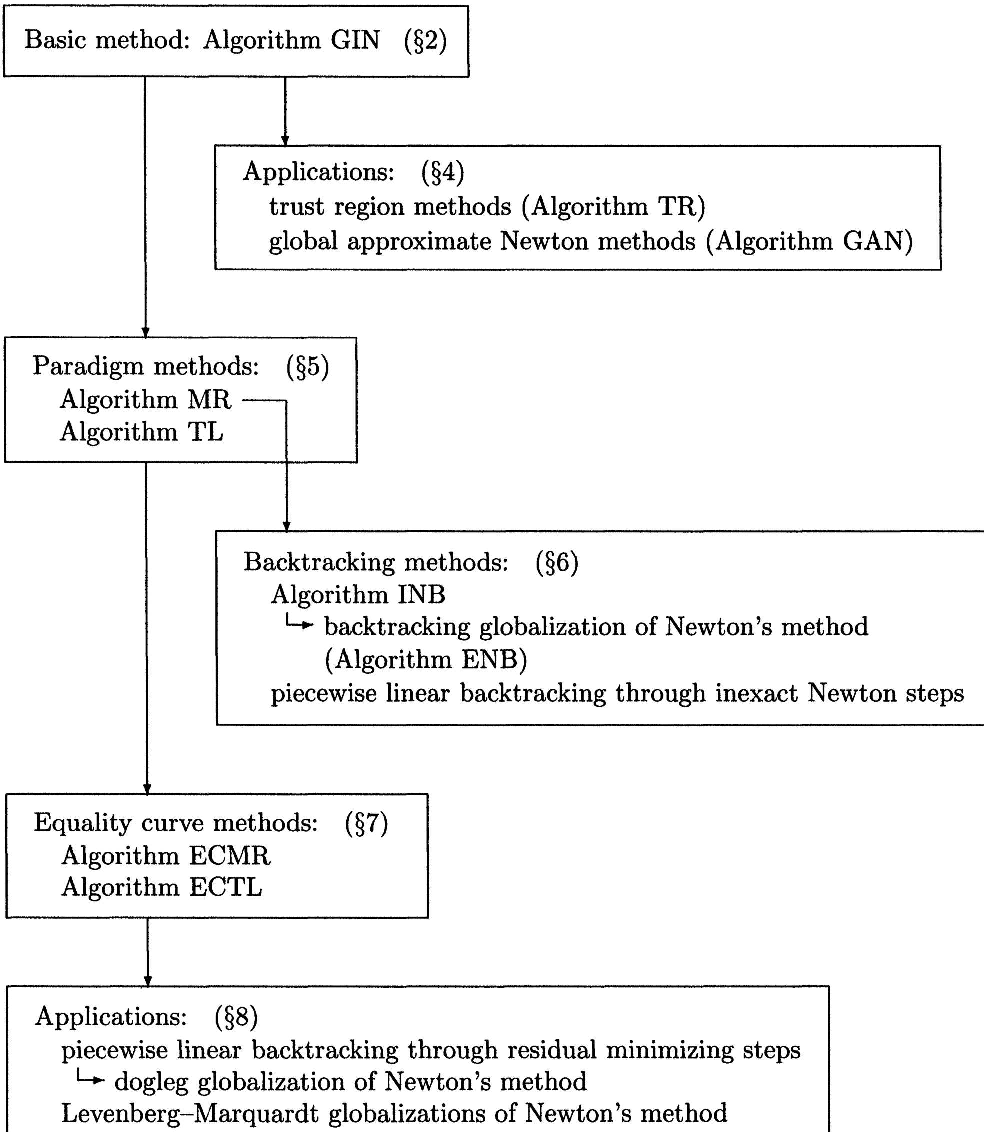

FIG. 1. *Organizational chart of algorithms and applications.*

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