Apply this general extension to

PDE Initial-Boundary Value Problem:

$$u_t = u_{xx}, \quad 0 \le x \le \pi, \quad 0 \le t < \infty$$
$$u(x,0) = f(x), \quad 0 \le x \le \pi \quad \text{(Initial Condition)}$$
$$u(0,t) = u(\pi,t) = 0, \quad 0 \le t < \infty \quad \text{(Boundary Conditions)}$$

This is a very simple model of temperature in a rod with insulated sides and ends held at temperature zero.

Suppose $f \in S_n = \{ \sum_{k=1}^{n} \alpha_k \sin kx : \alpha_k \in \mathbb{R}, 1 \le k \le n \} \subseteq C[0,\pi]$.

Recall from homework that $\{\sin kx\}_{k=1}^{n}$ is a basis for S_n and that, with respect to this basis, $T: S_n \to S_n$ defined by $T(f) = f''$ has matrix representation

$$\begin{pmatrix} -1 & & 0 \\ & -4 & \\ 0 & & \ddots \; -n^2 \end{pmatrix}, \text{ which is already diagonal.}$$

Then the eigenvalues and corresponding eigenvectors of T are $\lambda_k = -k^2$, $\sin kx$ for $1 \le k \le n$.

For $f \in S_n$, have $f(x) = \sum_{k=1}^{n} \alpha_k \sin kx$ for some α_k's.

Look for a solution u in S_n for each t. This will have form

$$u(x,t) = \sum_{k=1}^{n} \beta_k(t) \sin kx$$

Want $u(x,0) = f(x)$, so want $\beta_k(0) = \alpha_k$, $1 \le k \le n$.
Also want, for $0 \le x \le \pi$, $0 \le t < \infty$

$$u_t = u_{xx} \iff \sum_{k=1}^{n} \beta_k'(t) \sin kx = \sum_{k=1}^{n} -k^2 \beta_k(t) \sin kx,$$

This gives a decoupled system of ODE IVP's:

$$\left.\begin{aligned} \beta_k' &= -k^2 \beta_k \\ \beta_k(0) &= \alpha_k \end{aligned}\right\} k=1,\dots,n$$

$$\Leftrightarrow \quad \beta_k(t) = \alpha_k e^{-k^2 t}, \quad k=1,\dots,n.$$

Then the solution of the PDE IBVP is

$$u(x,t) = \sum_{k=1}^{n} \alpha_k e^{-k^2 t} \sin kx$$

This is a toy problem, but it tells us a lot, e.g., that the solution decays exponentially with a rate determined by the eigenvalues of the spatial operator w.r.t. the domain $[0,\pi]$. These things generalize.

What if $f \notin S_n$. Fourier analysis says that if $f \in C[0,\pi]$ and $f(0)=f(\pi)=0$, then

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \sin kx, \quad 0 \le x \le \pi,$$

where $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, and we can approximate f arbitrarily closely by $f_n(x) = \sum_{k=1}^{n} \alpha_k \sin kx$ by taking n sufficiently large. Moreover, as before, the solutions of the PDE IBVPs with ICs f and f_n are given by

$$u(x,t) = \sum_{k=1}^{\infty} \alpha_k e^{-k^2 t} \sin kx, \quad u_n(x,t) = \sum_{k=1}^{n} \alpha_k e^{-k^2 t} \sin kx,$$

and the error for $t \ge 0$ is

$$|u(x,t) - u_n(x,t)| = \left| \sum_{k=n+1}^{\infty} \alpha_k e^{-k^2} \sin kx \right| = O\left(e^{-(n+1)^2 t}\right),$$

so the error decays exponentially (faster than the solution).

Set $M\mathcal{S}_i = \{Ms : s \in \mathcal{S}_i\}$. This is a subspace of $\mathbb{C}^n$, and

$$A(M\mathcal{S}_i) = MJM^{-1}M\mathcal{S}_i = MJ\mathcal{S}_i \subseteq M\mathcal{S}_i,$$

so $M\mathcal{S}_i$ is an invariant subspace of A.

Work toward the Jordan Normal Form, First consider

<u>Nilpotent matrices.</u> $A \in \mathbb{C}^{n\times n}$ is <u>nilpotent with index ν</u> if $A^\nu = 0$ and $A^{\nu-1} \neq 0$.

<u>Ex.</u>: $A = \begin{pmatrix} 0&1&1\\ 0&0&1\\ 0&0&0\end{pmatrix}$, $A^2 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 0&0&0\end{pmatrix}$, $A^3 = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\end{pmatrix}$

so A is nilpotent with index 3.

Note 1: A nilpotent $\Rightarrow \sigma(A) = \{0\}$.

Why? If $Av = \lambda v$ for $\lambda \neq 0$ and $v \neq 0$, then $A^k v = \lambda^k v \neq 0$ for all $k \Rightarrow A$ is not nilpotent.

Note 2: A is nilpotent and diagonalizable $\Rightarrow A = 0$.

Why? If A is nilpotent and $A = M\Lambda M^{-1}$, then we must have $\Lambda = 0$ and $A = 0$.

It follows from Note 1 that the Jordan normal form for a nilpotent A is

$$A = MJM^{-1}, \quad J = \begin{pmatrix} 0 & x & & 0 \\ & \ddots & \ddots & \\ & & \ddots & x \\ 0 & & & 0 \end{pmatrix}$$

where each "x" is 0 or 1. Work toward this by developing a special basis of $\mathbb{C}^n$.

Suppose A is nilpotent, index ν.

Back to matrices. What if an $n \times n$ matrix A is not diagonalizable? The next best thing is the

<u>Jordan Normal Form</u>: Suppose A has k <u>distinct</u> eigenvalues. Then A can be written as

$$A = MJM^{-1}, \text{ where } J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix} \text{ and, for each } i,$$

$$J_i = \begin{pmatrix} \lambda_i & x & & 0 \\ & \ddots & \ddots & \\ & & \ddots & x \\ 0 & & & \lambda_i \end{pmatrix} \text{ and each "}x\text{" is either 0 or 1.}$$

See: (1) J is block diagonal an upper triangular.
(2) The <u>algebraic</u> multiplicity of λ_i is the number of rows (columns) of J_i, i.e., J_i is $m_i \times m_i$.
(3) The <u>geometric</u> multiplicity of λ_i is $1 +$ # zeros in the first superdiagonal of J_i.

(2)-(3) confirm that $k_i \le m_i$ always.

<u>Example</u>: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ This is already in Jordan normal form. Only eigenvalue is 1, with $m_i = 3$. Geometric multiplicity is $1+1=2$.

<u>Def.</u>: A subspace $\mathcal{S} \subseteq \mathcal{V}$ is invariant w.r.t. $T: \mathcal{V} \to \mathcal{V}$ if $T(\mathcal{S}) \subseteq \mathcal{S}$, i.e., $T(v) \in \mathcal{S}$ whenever $v \in \mathcal{S}$.

<u>Note</u>: If $\mathcal{S}$ is invariant w.r.t. T, then we can consider the <u>restriction</u> $T: \mathcal{S} \to \mathcal{S}$.

<u>See</u>: $J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix}$ has natural invariant subspaces.

Suppose $J_i \in \mathbb{C}^{m_i \times m_i}$. Set $\mathcal{S}_i = $ set of $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_1 \\ \vdots \\ \alpha_{m_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^n$,

where the zeros correspond to components outside the i^{th} block. Then $\mathcal{S}_i$ is an invariant subspace of J, i.e., $J(\mathcal{S}_i) \subseteq \mathcal{S}_i$.

Have $\mathbb{C}^n \supseteq \mathcal{R}(A) \supseteq \mathcal{R}(A^2) \supseteq \dots \supseteq \mathcal{R}(A^{\nu-1}) \supseteq \mathcal{R}(A^\nu) = \{0\}$.

Then

$$\underset{\mathcal{M}_0}{\mathcal{N}(A)} \supseteq \underset{\mathcal{M}_1}{\mathcal{N}(A) \cap \mathcal{R}(A)} \supseteq \underset{\mathcal{M}_2}{\mathcal{N}(A) \cap \mathcal{R}(A^2)} \supseteq \dots \supseteq \underset{\mathcal{M}_{\nu-1}}{\mathcal{N}(A) \cap \mathcal{R}(A^{\nu-1})} \supseteq \underset{\mathcal{M}_\nu}{\mathcal{N}(A) \cap \mathcal{R}(A^\nu)} = \{0\}.$$

Set $j_i = \dim \mathcal{M}_i$, $0 \le i \le \nu$. Note: $j_{\nu-1} \le j_{\nu-2} \le \dots \le j_1 \le j_0$.

Choose a basis $\{v_{11}, \dots, v_{1 j_{\nu-1}}\}$ of $\mathcal{M}_{\nu-1}$.

Since $\mathcal{M}_{\nu-1} \subseteq \mathcal{M}_{\nu-2}$, we can extend this to a basis $\{v_{11}, \dots, v_{1 j_{\nu-2}}\}$ of $\mathcal{M}_{\nu-2}$.

Continue to obtain bases $\{v_{11}, \dots, v_{1 j_{\nu-i}}\}$ of $\mathcal{M}_{\nu-i}$, $1 \le i \le \nu$.
Note: $A v_{1j} = 0$ for $j = 1, \dots, j_0$ since each $\mathcal{M}_j$ is in $\mathcal{N}(A)$.

Now construct "Jordan chains" from these basis vectors.
Start with basis $\{v_{11}, \dots, v_{1 j_{\nu-1}}\}$ of $\mathcal{M}_{\nu-1}$.

For $1 \le j \le j_{\nu-1}$, $v_{1j} \in \mathcal{M}_{\nu-1} = \mathcal{N}(A) \cap \mathcal{R}(A^{\nu-1})$, so

$$v_{1j} = A^{\nu-1} v \quad \text{for some } v \in \mathbb{C}^n.$$

Set $v_{ij} = A^{\nu-i} v$ for $i = 1, \dots, \nu$. (Note: $v_{\nu j} = v$.)

Now do a similar thing with the remaining vectors in the basis for $\mathcal{M}_{\nu-2}$.

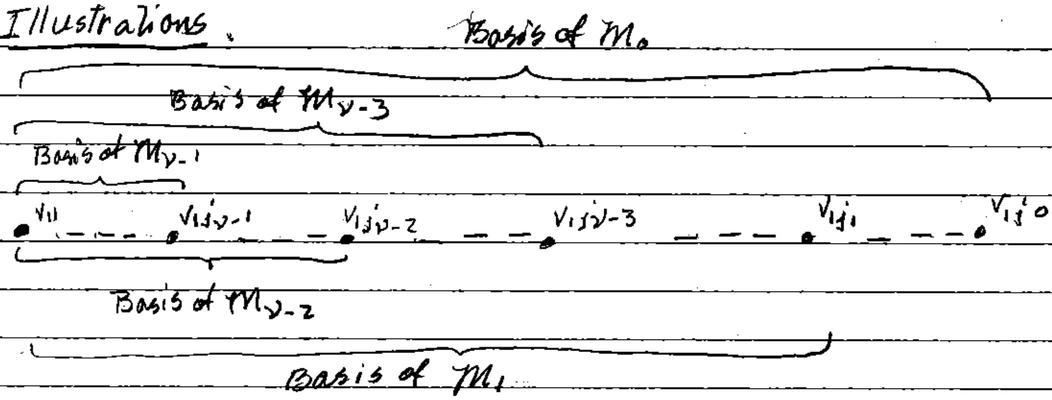
For $j_{\nu-1} < j \le j_{\nu-2}$, $v_{1j} \in \mathcal{M}_{\nu-2} = \mathcal{N}(A) \cap \mathcal{R}(A^{\nu-2})$, so

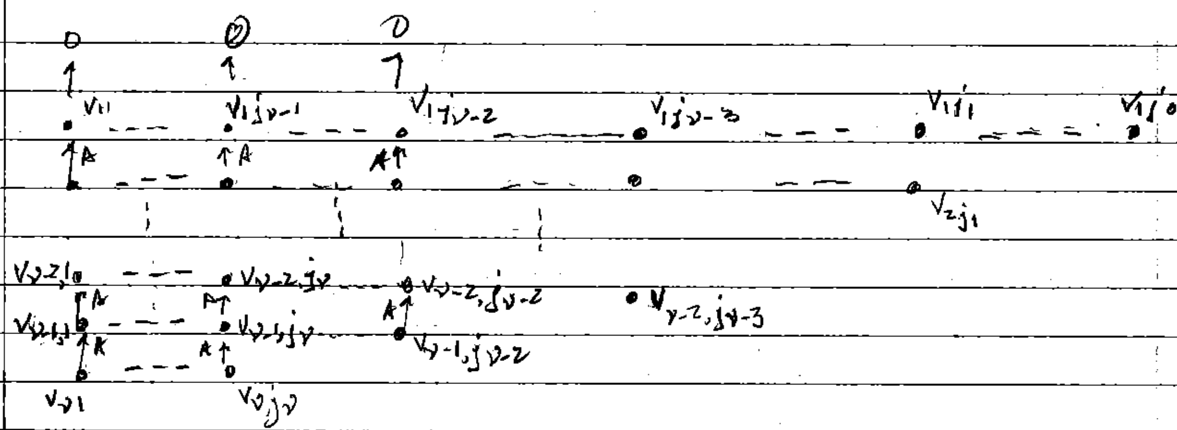
$$v_{1j} = A^{\nu-2} v \quad \text{for some } v \in \mathbb{C}^n.$$

Set $v_{ij} = A^{\nu-1-i} v$, $i = 1, \dots, \nu-1$. (Note: $v_{\nu-1, j} = v$.)

Continue. At the final step, the remaining vectors in the basis for $\mathcal{M}_0$ are vectors v_{1j}, $j_1 < j \le j_0$, and there is nothing to do.

Illustrations.





At the end, we have $\{v_{ij}\}_{\substack{1 \le i \le \nu \\ 1 \le j \le j_{i-1}}}$. Note: $Av_{ij} = \begin{cases} v_{i-1,j} & \text{if } i > 1 \\ 0 & \text{if } i = 1 \end{cases}$

Claim: This is a basis for $\mathbb{C}^n$.

(1) Linearly independent.

Suppose $\sum_{i=1}^{\nu} \sum_{j=1}^{j_{i-1}} \alpha_{ij} v_{ij} = 0$. Then

$$0 = A^{\nu-1}\left(\sum_{i=1}^{\nu-1} \sum_{j=1}^{j_{i-1}} \alpha_{ij} v_{ij} + \sum_{j=1}^{j_{\nu-1}} \alpha_{\nu j} v_{\nu j} \right)$$

$$= \sum_{j=1}^{j_{\nu-1}} \alpha_{\nu j} v_{1j}$$

$\Rightarrow \alpha_{\nu j} = 0$, $1 \le j \le j_{\nu-1}$, since $v_{11}, \ldots, v_{1 j_{\nu-1}}$ are linearly independent.

Then we have $\sum_{i=1}^{\nu-1} \sum_{j=1}^{j_{i-1}} \alpha_{ij} v_{ij} = 0$, so

$$0 = A^{\nu-2}\left(\sum_{i=1}^{\nu-2} \sum_{j=1}^{j_{i-1}} \alpha_{ij} v_{ij} + \sum_{j=1}^{j_{\nu-2}} \alpha_{\nu-1,j} v_{\nu-1,j} \right)$$

$$= \sum_{j=1}^{j_{\nu-2}} \alpha_{\nu-1,j} v_{1j}$$

$\Rightarrow \alpha_{\nu-1,j} = 0$, $1 \le j \le j_{\nu-2}$, since $v_{11}, \ldots, v_{1 j_{\nu-2}}$ are linearly independent.

Continue to show that all α_{ij} are zero.

(2) Spans. Suppose $v \in \mathbb{C}^n$. We have that

$$A^{\nu-1} v \in \mathcal{R}(A^{\nu-1}) \cap \mathcal{N}(A) = \mathcal{M}_{\nu-1}$$

$$\Rightarrow A^{\nu-1} v = \sum_{j=1}^{j_{\nu-1}} \alpha_{\nu j} v_{1j}$$

Set $u_{\nu-1} = \sum_{j=1}^{j_{\nu-1}} \alpha_{\nu j} v_{\nu j}$. Then $A^{\nu-2}(v-u_{\nu-1}) \in \mathcal{R}(A^{\nu-2})$

and $A^{\nu-1}(v-u_{\nu-1}) = A^{\nu-1}v - \sum_{1}^{j_{\nu-1}} \alpha_{\nu j} \overbrace{A^{\nu-1}v_{\nu j}}^{v_{1j}} = 0$, so $A^{\nu-2}(v-u_{\nu-1}) \in \mathcal{N}(A)$

so $A^{\nu-2}(v-u_{\nu-1}) \in \mathcal{M}_{\nu-2}$, and $A^{\nu-2}(v-u_{\nu-1}) = \sum_{j=1}^{j_{\nu-2}} \alpha_{\nu-1,j} v_{1j}$.

Set $u_{\nu-2} = \sum_{j=1}^{j_{\nu-2}} \alpha_{\nu-1,j} v_{\nu-1,j}$. Then, as before,

$A^{\nu-3}(v-u_{\nu-1}-u_{\nu-2}) \in \mathcal{R}(A^{\nu-3}) \cap \mathcal{N}(A) = \mathcal{M}_{\nu-3}$,

so $A^{\nu-3}(v-u_{\nu-1}-u_{\nu-2}) = \sum_{j=1}^{j_{\nu-3}} \alpha_{\nu-2,j} v_{1j}$.

Set $u_{\nu-3} = \sum_{j=1}^{j_{\nu-3}} \alpha_{\nu-3,j} v_{\nu-3,j}$.

Continue to get

$$v - u_{\nu-1} - \dots - u_1 = u_0 \in \mathcal{N}(A) = \mathcal{M}_0, \quad u_0 = \sum_{j=1}^{j_0} \alpha_{1j} v_{1j}$$

Then $v = u_{\nu-1} + \dots + u_1 + u_0 = \sum_{i=1}^{\nu} \sum_{j=1}^{j_{i-1}} \alpha_{ij} v_{ij}$.

What is the representation of A with respect to this basis? Order by

$$\underbrace{v_{\nu 1} \dots v_{11}\ v_{\nu 2} \dots v_{12} \dots v_{1 j_{\nu-1}}}_{\text{for } \mathcal{M}_{\nu-1}}\ \underbrace{v_{\nu-1,1} \dots v_{1 j_{\nu-2}}}_{\text{for } \mathcal{M}_{\nu-2}}\ v_{\nu-2,1} \cdots\cdots v_{1 j_0}$$

In each Jordan chain $v_{\nu-i+1,j} \dots v_{1j}$, we have

$$A v_{\ell j} = \begin{cases} v_{\ell-1,j} & \text{if } \ell > 1 \\ 0 & \text{if } \ell = 1 \end{cases}.$$

Then $\operatorname{span}\{v_{\nu-i+1,j}, \dots, v_{1j}\}$ is an <u>invariant subspace</u> of A. With respect to the basis $\{v_{\nu-1,j}, \dots, v_{1j}\}$, the restriction of A to this subspace has the representation

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \sim (\nu - i + 1) \times (\nu - i + 1).$$

Continue this process to include all Jordan chains. Conclude that with respect to the ordered basis above, A has the representation

$$J = \begin{pmatrix} 0 & x & & 0 \\ & \ddots & \ddots & \\ & & \ddots & x \\ 0 & & & 0 \end{pmatrix}, \text{ where each "x" is } 0 \text{ or } 1.$$

Then $A = MJM^{-1}$ for an appropriate M, and this is the Jordan form for a nilpotent matrix.

For general $A \in \mathbb{C}^{n \times n}$, let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues.

Set $A_1 = A - \lambda_1 I$ and $S_1 = \{ v \in \mathbb{C}^n : A_1^p v = 0 \text{ for some } p \}$.
Note: S_1 is an invariant subspace for A_1.
Note: If $A^{p-1} v \neq 0$ and $A^p = 0$ for $p \geq 1$, then $v, Av, \ldots, A^{p-1}v$ are linearly independent.

Why? Suppose $\sum_{i=0}^{p-1} \alpha_i A^i v = 0$. Then

$$0 = A^{p-1}\left(\sum_{i=0}^{p-1} \alpha_i A^i v\right) = \alpha_0 A^{p-1} v + \sum_{i=1}^{p-1} \alpha_i A^{p-1+i}$$

$$= \alpha_0 A^{p-1} v$$

$\Rightarrow \alpha_0 = 0$ and $0 = \sum_{i=1}^{p-1} \alpha_i A^i v$. Apply A^{p-2}, etc., and continue to show $\alpha_0 = \cdots = \alpha_{p-1} = 0$.

It follows that, for $v \in S_1$, $A_1^p v = 0$ for some $p \leq n$.

Set $\nu = \min\{p : A_1^p v = 0 \text{ for } \underline{all}\ v \in \mathcal{S}_1\}$.
Then A_1 is nilpotent with index ν on the invariant subspace $\mathcal{S}_1$, and there is a basis for $\mathcal{S}_1$ with respect to which A_1 has matrix representation

$$\begin{pmatrix} 0 & x & & 0 \\ & \ddots & \ddots & \\ & & \ddots & x \\ 0 & & & 0 \end{pmatrix} \text{ where each "x" is 0 or 1.}$$

Since $A = A_1 + \lambda_1 I$, $\mathcal{S}_1$ is also an invariant subspace for A, and the representation of A with respect to the basis is

$$\begin{pmatrix} \lambda_1 & x & & 0 \\ & \ddots & \ddots & \\ & & \ddots & x \\ 0 & & & \lambda_1 \end{pmatrix}.$$

Continue this process for $\lambda_2, \dots, \lambda_k$ to obtain a basis for $\mathbb{C}^n$ with respect to which A has representation

$$J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & x & & 0 \\ & \ddots & \ddots & \\ & & \ddots & x \\ 0 & & & \lambda_i \end{pmatrix} \text{ where each "x" is 0 or 1.}$$

Then $A = MJM^{-1}$ for an appropriate M, and this is the Jordan form for A.

__Remarks.__

(1) We can alternatively write the Jordan form as $A = MJM^{-1}$, where
$$J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_\ell \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$
and $\lambda_1, \dots, \lambda_\ell$ are not necessarily distinct.

(2) In each J_i, we can replace the 1's above the diagonal by any nonzero number. In particular, given $\epsilon > 0$, we can take
$$J_i = \begin{pmatrix} \lambda_i & \epsilon & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \epsilon \\ 0 & & & \lambda_i \end{pmatrix} \text{ for any/all } i.$$

How? Suppose J_i is $p \times p$. Set $L_i = \begin{pmatrix} \epsilon^p & & 0 \\ & \epsilon^{p-1} & \\ & \ddots & \\ 0 & & \epsilon \end{pmatrix}$.

Then

$$L_i J_i L_i^{-1} = \begin{pmatrix} \lambda_i & \epsilon & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \epsilon \\ 0 & & & \lambda_i \end{pmatrix}.$$

Do this for each i and set

$$L = \begin{pmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_\ell \end{pmatrix}$$

Then

$$A = MJM^{-1} = \underbrace{ML^{-1}}_{\hat{M}} \underbrace{LJL^{-1}}_{\hat{J}} \underbrace{LM^{-1}}_{\hat{M}^{-1}} = \hat{M}\hat{J}\hat{M}^{-1}.$$