Determinants

Begin by developing the determinant $\det A$ of a (square) $A \in \mathbb{R}^{n\times n}$. Then proceed to develop the determinant a a general linear $T: \mathcal{V} \to \mathcal{W}$ between finite-dimensional vector spaces of the same dimension.

First, a <u>permutation</u> of $\{1, \ldots, n\}$ is just a rearrangement or reordering, expressed by $j \to \pi(j)$ for $1 \le j \le n$, where $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is 1-1 and onto. Can think of as $(1, 2, \ldots, n) \to (\pi(1), \pi(2), \ldots, \pi(n))$.

<u>Ex.</u>: $(1,2,3,4) \to (3,1,4,2)$, so $\pi(1) = 3$, $\pi(2) = 1$, etc.

An <u>elementary permutation</u> is one that interchanges (swaps) two entries and leaves all others unchanged.

<u>Ex.</u>: $(1,2,3,4) \to (1,3,2,4)$.

<u>Fact</u>: Every permutation π of $\{1, \ldots, n\}$ can be written as a finite product (composition) of elementary permutations $\pi = \pi_k \circ \pi_{k-1} \circ \cdots \circ \pi_1$.

<u>Ex</u>: $(1,2,3,4) \xrightarrow{\pi} (3,1,4,2)$ is equivalent to

$$(1,2,3,4) \xrightarrow{\pi_1} (2,1,3,4) \xrightarrow{\pi_2} (2,1,4,3) \xrightarrow{\pi_3} (3,1,4,2).$$

<u>and</u>

$$(1,2,3,4) \xrightarrow{\pi_1} (1,2,4,3) \xrightarrow{\pi_2} (2,1,4,3) \xrightarrow{\pi_3} (3,1,4,2)$$

So a decomposition $\pi = \pi_k \circ \pi_{k-1} \circ \dots \circ \pi_1$ is not unique. However, ...

Fact: For a given permutation π, the number of elementary transformations in a decomposition $\pi = \pi_k \circ \dots \circ \pi_1$ is either even or odd for all such permutations.

This allows us to define the sign or parity of a permutation π. If $\pi = \pi_k \circ \dots \circ \pi_1$ for elementary permutations $\pi_1, \dots, \pi_k$, then the sign (parity) of π is:

$$\sigma(\pi) = \begin{cases} +1, & \text{if } k \text{ is even} \\ -1, & \text{if } k \text{ is odd} \end{cases}$$

Now determinants.

Preliminary remarks: Wonderful historical summary in Meyer (Sec. 6.1), showing roots of determinants going back ≈ 2000 years in China and over 300 years in Germany and Japan, ultimately becoming the major tool for analyzing and solving linear systems in the late 19th century. Now, though, ...

"Determinants are much further from the center of linear algebra than they were a hundred years ago."
— Gil Strang

Today, they are useful for some theoretical purposes but essentially never for computation, although there are efficient, reliable ways of computing them.

There are several ways to define determinants. All are equivalent, and all, at some point, involve something very messy.

Here, we'll start with the mess up front in the definition.

Def. 4.1: The <u>determinant</u> of $A \in \mathbb{R}^{n\times n}$ is

$$\det A = \sum_{\pi} \sigma(\pi)\, a_{1\pi(1)}\, a_{2\pi(2)} \cdots a_{n\pi(n)} \qquad (4.1)$$

where a_{ij} is the ij^{th} entry in A and the sum is over all permutations of $\{1,\dots,n\}$.

Sometimes denote $\det A$ by $|A|$.

<u>Familiar examples.</u>

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Permutations are $(1,2)\to(1,2) \Rightarrow +ad$
$(1,2)\to(2,1) \Rightarrow -bc$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + dhc + gbf - gec - ahf - dbi$$

Permutations are $(1,2,3)\to(1,2,3) \Rightarrow +aei$
" $\to(2,3,1) \Rightarrow +dhc$
" $\to(3,1,2) \Rightarrow +gbf$
" $\to(3,2,1) \Rightarrow -gec$
" $\to(1,3,2) \Rightarrow -ahf$
" $\to(2,1,3) \Rightarrow -dbi$

Note: The defining formula should <u>never</u> be used for computation when $n > 3$. <u>Never</u> use Cramer's Rule to solve a linear system when $n > 3$.

Despite its practical uselessness, (4.1) gives nice insights. Will use it to develop some basic properties that are key to understanding and using determinants.

P1: If A is triangular, then $\det A = a_{11} a_{22} \cdots a_{nn}$.

Cor. 4.2: $\det I = 1$

The proof relies on a more generally useful observation: each term $\sigma(\pi) a_{1\pi(1)} \cdots a_{n\pi(n)}$ in the defining sum involves exactly one entry from each row and each column of A.

Pf: Suppose A is, say, lower triangular. Then the only (possibly) non-zero entry in the 1st row of A is a_{11}. So all terms in the defining sum vanish except those for which $\pi(1) = 1$. Then the sum becomes

$$\det A = \sum_{\{\pi : \pi(1)=1\}} \sigma(\pi)\, a_{11}\, a_{2\pi(2)} \cdots a_{n\pi(n)}$$

Note that all π in this sum must have $2 \le \pi(2) \le n$, since $\pi(1) = 1$. But for all such π, $a_{2\pi(2)} = 0$ whenever $\pi(2) > 2$. Then the sum reduces to

$$\det A = \sum_{\{\pi : \pi(1)=1,\, \pi(2)=2\}} \sigma(\pi)\, a_{11}\, a_{22}\, a_{3\pi(3)} \cdots a_{n\pi(n)}$$

Continuing, we ultimately obtain $\det A = a_{11} a_{22} \cdots a_{nn}$. Note that the sign is + corresponding to the identity permutation.

P2: $\det A = \det A^T$.

Pf.: $$\sum_{\pi} \sigma(\pi)\, a_{1\pi(1)} \cdots a_{n\pi(n)} = \sum_{\pi} \sigma(\pi)\, a_{\pi^{-1}(1)1} \cdots a_{\pi^{-1}(n)n}$$

$$= \sum_{\pi} \sigma(\pi)\, a_{\pi(1)1} \cdots a_{\pi(n)n} = \det A^T.$$

P3: Exchanging two rows or columns of A changes sign of the determinant.

Pf: Suppose $\tilde{A}$ is obtained by exchanging rows i and j of A. Then

$$\det \tilde{A} = \sum_{\pi} \sigma(\pi)\, a_{1\pi(1)} \cdots \overset{i\text{th}}{a_{j\pi(i)}} \cdots \overset{j\text{th}}{a_{i\pi(j)}} \cdots a_{n\pi(n)}$$

$$= \sum_{\pi} \sigma(\pi)\, a_{1\pi(1)} \cdots \overset{i\text{th}}{a_{i\pi(j)}} \cdots \overset{j\text{th}}{a_{j\pi(i)}} \cdots a_{n\pi(n)}$$

$$= \sum_{\pi} -\sigma(\pi)\, a_{1\pi(1)} \cdots a_{i\pi(i)} \cdots a_{j\pi(j)} \cdots a_{n\pi(n)}$$

$$= -\det A.$$

Now let $\hat{A}$ be obtained by exchanging two colums of A. Since this corresponds to exchanging two rows of A^T, we have

$$\det \hat{A} = \det \hat{A}^T = -\det A^T = -\det A.$$

$\boxed{\text{Cor. 4.3: If } A \text{ has two equal rows or columns, then } \det A = 0.}$

P4: Multiplying a row or column of A by a scalar α multiplies the determinant by α.

Pf: Suppose $\hat{A}$ is obtained by multiplying the ith row of A by α. Then

$$\det \hat{A} = \sum_{\pi} \sigma(\pi)\, a_{1\pi(1)} \cdots (\alpha\, a_{i\pi(i)}) \cdots a_{n\pi(n)}$$

$$= \alpha \left\{ \sum_{\pi} \sigma(\pi)\, a_{1\pi(1)} \cdots a_{i\pi(i)} \cdots a_{n\pi(n)} \cdots \right.$$

$$= \det A$$

Similarly for columns.

<u>Cor 4.4.</u> If A has an all-zero row or column, then $\det A = 0$.

<u>P5:</u> Adding a scalar multiple of one row or column to another does not change the determinant.

<u>Pf:</u> Suppose $\hat{A}$ is obtained by adding $\alpha \times$ (j^{th} row) to the i^{th} row. Then

$$
\begin{aligned}
\det \hat{A} &= \sum \sigma(\pi)\, a_{1\pi(1)} \cdots (a_{i\pi(i)} + \alpha\, a_{j\pi(i)}) \cdots a_{j\pi(j)} \cdots a_{n\pi(n)} \\
&= \sum \sigma(\pi)\, a_{1\pi(1)} \cdots a_{i\pi(i)} \cdots a_{j\pi(j)} \cdots a_{n\pi(n)} \\
&\quad + \alpha \sum \sigma(\pi)\, a_{1\pi(1)} \cdots a_{j\pi(i)} \cdots a_{j\pi(j)} \cdots a_{n\pi(n)} \\
&= \det A + \alpha \cdot 0 = \det A
\end{aligned}
$$

Since the second sum is the determinant of a matrix with the same i^{th} & j^{th} rows.
Similarly for columns.

<u>Cor 4.5.</u> Adding a linear combination of other rows (columns) to a row (column) does not change the value of the determinant.

<u>Cor 4.6</u> If one row (column) of A is a linear combination of other rows (columns), then $\det A = 0$.

Before prescribing the next two properties, need a short digression on elimination.

Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular.

Step 1: Exchange 1^{st} row with a row below if necessary to obtain a nonzero "pivot element" in upper left. This is equivalent to $A \to P_1 A$, where

$$P_1 = \begin{cases} I, & \text{if } a_{11} \text{ is nonzero} \\ \text{obtained from } I \text{ by exchanging } 1^{st} \text{ and } j^{th} & \\ \quad \text{rows for some } j > 1. & \end{cases}$$

Note: Since A is nonsingular, at least one entry in 1^{st} col^{n} is nonzero.

Note: $\det P_1 = \pm 1$ and $\det P_1 A = \pm \det A = \det P_1 \det A$.

For convenience, denote entries in $P_1 A$ by a_{ij}.

Set $M_1 = \begin{pmatrix} 1 & & \\ -m_{21} & 1 & 0 \\ \vdots & & \ddots \\ -m_{n1} & 0 & 1 \end{pmatrix}$, $\quad m_{i1} = \dfrac{a_{i1}}{a_{11}}$

Then $\quad M_1 P_1 A = \begin{pmatrix} a_{11} & x & — \\ 0 & x & — \\ \vdots & \vdots & — \\ 0 & x & — \end{pmatrix}$

Note: $\det M_1 = 1$, and $\det M_1 P_1 A = \det P_1 A = \det P_1 \det A$,

so
$$\det M_1 P_1 A = \det M_1 \det P_1 \det A = \det P_1 \det A,$$

Also, $M_1 P_1 A$ is nonsingular since M_1, P_1, and A are.

Step k: Have $M_{k-1} P_{k-1} \cdots M_1 P_1 A = \begin{pmatrix} a_{11} & & — \\ & \ddots \; a_{kk} & — \\ 0 & \vdots & — \\ & a_{nk} & — \end{pmatrix}$ nonsingular with

$$\det(M_{k-1} P_{k-1} \cdots M_1 P_1 A) = \det M_{k-1} \det P_{k-1} \cdots \det M_1 \det P_1 \det A$$
$$= \det P_{k-1} \cdots \det P_1 \det A$$

Determine $P_k = \begin{cases} I, \text{ if } a_{kk} \neq 0 \\ \text{obtained by exchanging } k^{th} \text{ and } j^{th} \text{ rows for some } j > k. \end{cases}$

so that $P_k M_{k-1} P_{k-1} - M_1 P_1 A$ has a nonzero pivot element a_{kk}.

Set $M_k = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ 0 & & -m_{k+1,k} & \ddots & \\ & & -m_{n,k} & & 1 \end{pmatrix}$, $m_{ik} = \dfrac{a_{ik}}{a_{kk}}$

Then $M_k P_k - M_1 P_1 A = \begin{pmatrix} a_{11} & & & \\ & \ddots & & \\ & & a_{kk} & \\ 0 & & 0 & x \\ & & 0 & x \end{pmatrix}$

End with

$$M_{n-1} P_{n-1} -- M_1 P_1 A = U$$

where U is upper-triangular with non-zero diagonal entries (since A is nonsingular), and

$$\det U = \det P_{n-1} - \det P_1 \det A$$

End of digression. Back to determinant properties.

P6. A is nonsingular $\Leftrightarrow \det A \neq 0$; equivalently, A is singular $\Leftrightarrow \det A = 0$.

Pf. Suppose A is singular. Then A has column rank $< n$, i.e., the columns of A are linearly dependent. Then (at least) one column of A can be written as a linear combination of the others, and it follows the second corollary of P4 (cor. 4.6) that $\det A = 0$.
Now suppose A is nonsingular. Then, as above, we have

$$0 \neq \det U = \det M_{n-1} \det P_{n-1} - \det M_1 \det P_1 \det A$$

Since $\det M_i = 1$ and $\det P_i = \pm 1$, it follows that $\det A \neq 0$.

P7. $\det AB = \det A \det B$ for $A, B \in \mathbb{R}^{n\times n}$

Pf.: By Prop. 3.11, have that AB is singular $\Leftrightarrow$ either A or B is singular. In view of P5, the property clearly holds in this case.

Suppose that both A and B are nonsingular. First, assume that A is upper triangular;

$$A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

and denote the rows of B by $b_i = (b_{i1}, \ldots, b_{in})$ for $1 \le i \le n$. Then

$$AB = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j} b_j \\ \sum_{j=2}^{n} a_{2j} b_j \\ \vdots \\ \sum_{j=n-1}^{n} a_{n-1,j} b_j \\ a_n b_n \end{pmatrix} = \begin{pmatrix} a_{11} b_1 + \sum_{2}^{n} a_{ij} b_j \\ a_{22} b_2 + \sum_{3}^{n} a_{ij} b_j \\ \vdots \\ a_{n-1,n-1} b_{n-1} + a_{n-1,n} b_n \\ a_{nn} b_n \end{pmatrix}$$

and, repeatedly applying P4 and Cor. 4.4 gives

$$\det AB = a_{nn} \det \begin{pmatrix} a_{11} b_1 + \sum a_{ij} b_j \\ \vdots \\ a_{n-1,n-1} b_{n-1} + a_{n-1,n} b_n \\ b_n \end{pmatrix}$$

$$\vdots$$

$$= a_{11} \cdots a_{nn} \det \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \det A \det B$$

For general nonsingular A, can write

$$M_{n-1} P_{n-1} \cdots M_1 P_1 A = U$$

Then

$$A = P_1^{-1} M_1^{-1} \cdots P_{n-1}^{-1} M_{n-1}^{-1} U$$

and

$$\det A = \det P_1^{-1} \cdots \det P_{n-1}^{-1} \det U$$

Then

$$\det AB = \det (P_1^{-1} M_1^{-1} \cdots P_{n-1}^{-1} M_{n-1}^{-1} U B)$$

$$= \det P_1^{-1} \cdots \det P_{n-1}^{-1} \det UB$$

$$= \det P_1^{-1} \cdots \det P_{n-1}^{-1} \det U \det B$$

$$= \det A \det B$$