Look at some examples.

Ex. 1: $\mathbb{R}^n = \{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \text{ for } i = 1, \dots, n \}$.

"Natural" basis: $\{e_1, \dots, e_n\}$, where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ place.

See: $\sum \alpha_i e_i = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0 \Leftrightarrow \alpha_1 = \dots = \alpha_n = 0 \Rightarrow$ linearly independent

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum x_i e_i \Rightarrow \text{spans}$$

So $\dim \mathbb{R}^n = n$.

Ex. 2: $\mathbb{R}^{m \times n}$ = $m \times n$ real matrices

"Natural" basis: $\{ E_{ij} : 1 \le i \le m, 1 \le j \le n \}$, where the ij^{th} entry of E_{ij} is 1 and all other entries are 0.

So $\dim \mathbb{R}^{m \times n} = mn$.

Ex. 3: $\mathcal{P}_n$ = set of polynomials with real coefficients and degree $\le n$.

"Natural" basis: $\{1, x, x^2, \dots, x^n\}$.

So $\dim \mathcal{P}_n = n+1$.

Note: $\mathcal{P}$ = set of all polynomials with real coefficients is infinite-dimensional.

Ex. 4: $C[0,1]$ is infinite-dimensional.

Theorem 1.1 and Lemma 1.2 are very powerful. Draw some corollaries.

Suppose $\mathcal{V}$ is a finite-dimensional vector space, dimension n, so $\mathcal{V}$ has a basis with n vectors.

Cor. 2.1. A basis for $\mathcal{V}$ is a maximal linearly independent set, i.e., every set of more than n vectors must be linearly dependent.

Pf: Since $\mathcal{V}$ has a basis of n vectors, it has a spanning set of n vectors. It follows from Lemma 1.2 that no linearly independent set in $\mathcal{V}$ can have more than n vectors.

Cor 2.2. A basis for $\mathcal{V}$ is a minimal spanning set, i.e., a set with fewer than n vectors cannot span.

Pf: Since $\mathcal{V}$ has a basis of n vectors, it has a linearly independent set of n vectors. Then Lemma 1.2 implies that every spanning set must have at least n vectors.

Prove a small but useful lemma.

Lemma 2.3. If $v_1, \ldots, v_k$ are linearly independent and $v \notin \text{span}\{v_1, \ldots, v_k\}$, then $\{v_1, \ldots, v_k, v\}$ is linearly independent.

Pf.: Suppose $\sum_{i=1}^{k} \alpha_i v_i + \beta v = 0$. If $\beta \neq 0$, then

$$v = -\sum_{i=1}^{k} \frac{\alpha_i}{\beta} v_i \in \text{span}\{v_1, \ldots, v_k\};$$

so $\beta = 0$ and $\sum_{i=1}^{k} \alpha_i v_i = 0$. Since $v_1, \ldots, v_k$ are linearly independent, it follows that $\alpha_1 = \ldots = \alpha_k = 0$

can now obtain further nice results.

Prop. 2.4. Every linearly independent set of n vectors is a basis.

Pf.: Suppose $v_1, \dots, v_n$ are linearly independent. If span $\{v_1, \dots, v_n\} \neq \mathcal{V}$, then Lemma 2.3 would imply that $\exists\ v \in \mathcal{V}$ such that $\{v_1, \dots, v_n, v\}$ is linearly independent, which contradicts Cor. 2.1.

Another useful lemma...

Lemma 2.5. If $\mathcal{X}_m = \{x_1, \dots, x_m\}$ is linearly dependent, then there is a subset $\mathcal{X}_{m-1} \subseteq \mathcal{X}_m$ with $m-1$ vectors such that span $\mathcal{X}_{m-1}$ = span $\mathcal{X}_m$.

Pf.: Since $\mathcal{X}_m$ is linearly dependent, we have $\sum \alpha_i x_i = 0$ with $\alpha_j \neq 0$ for some j, $1 \leq j \leq m$. Then $x_j = \sum_{i \neq j} -\frac{\alpha_i}{\alpha_j} x_i$. Let $\mathcal{X}_{m-1}$ be the set obtained by removing x_j from $\mathcal{X}_m$.

Claim: span $\mathcal{X}_{m-1}$ = span $\mathcal{X}_m$.

Why? Suppose $v = \sum_{i=1}^{m} \beta_i v_i$. Then, as in the proof of Lemma 1.2,

$$v = \sum_{i \neq j} \beta_i v_i + \beta_j \left(\sum_{i \neq j} -\frac{\alpha_i}{\alpha_j} x_i \right) = \sum_{i \neq j} \left(\beta_i - \frac{\beta_j \alpha_i}{\alpha_j} \right) x_i .$$

Prop. 2.6. Every spanning set of n vectors is a basis.

Pf.: Suppose span $\{v_1, \dots, v_n\} = \mathcal{V}$. If $v_1, \dots, v_n$ are not linearly independent, then Lemma 2.5 would imply that there is a spanning set of $n-1$ vectors, which contradicts Cor. 2.2.

Prop. 2.7. Every set of linearly independent vectors can be expanded to a basis.

Pf. Suppose $\{v_1, -, v_k\}$ is a linearly independent set. If $k = n$, then $\{v_1, -, v_k\}$ is already a basis by Prop. 2.4, so suppose $k < n$. Then Cor. 2.2 implies that span $\{v_1, -, v_k\} \neq \mathcal{V}$, and it follows from Lemma 2.3 that $\exists\, v_{k+1} \in \mathcal{V}$ such that $\{v_1, -, v_k, v_{k+1}\}$ is linearly independent. Continuing in this way, we ultimately obtain linearly independent $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$, which, by Prop. 2.4, is a basis.

Prop. 2.8. Every (finite) spanning set contains a basis.

Pf.: Suppose $\mathcal{X}_m = \{x_1, -, x_m\}$ is a spanning set. If $m = n$, then $\{x_1, -, x_m\}$ is already a basis by Prop. 2.6, so suppose $m > n$. Then $x_1, -, x_m$ must be linearly dependent by Cor. 2.1, and it follows from Lemma 2.5 that there is a subset $\mathcal{X}_{m-1} \subseteq \mathcal{X}_m$ that is a spanning set with $m-1$ vectors. Continuing in this way, we ultimately obtain a spanning set of n vectors, which, by Prop. 2.6, is a basis.

Bases and dimensions of subspaces.

A subspace $\mathcal{S}$ of a vector space $\mathcal{V}$ is itself a vector space, so all results concerning basis and dimension apply to $\mathcal{S}$ as well as to $\mathcal{V}$. Since a basis for $\mathcal{S}$ is linearly independent in $\mathcal{V}$ as well as in $\mathcal{S}$, it follows immediately from Cor. 2.1 that $\dim \mathcal{S} \leq \dim \mathcal{V}$, with $\dim \mathcal{S} = \dim \mathcal{V} \Leftrightarrow \mathcal{S} = \mathcal{V}$.

The following very useful result is easily obtained.

Prop. 2.9 A basis $\{v_1, -, v_k\}$ for $\mathcal{S}$ can be expanded to a basis $\{v_1, -, v_k, v_{k+1}, -, v_n\}$ for $\mathcal{V}$.

Pf.: Since $\{v_1, -, v_k\}$ is linearly independent in $\mathcal{V}$ as well as in $\mathcal{S}$, the proposition follows immediately from Prop. 2.7.

Linear transformations and matrices.

Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector spaces over $\mathcal{F}$.

Def.: A linear transformation from $\mathcal{V}$ to $\mathcal{W}$ is a map $T: \mathcal{V} \to \mathcal{W}$ that satisfies $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for every $x, y \in \mathcal{V}$ and every $\alpha, \beta \in \mathcal{F}$.

Look at some examples.

Ex. 1: A matrix $A \in \mathbb{R}^{m \times n}$ naturally defines a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(x) = Ax$ for $x \in \mathbb{R}^n$. Since $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for all $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, this is clearly linear.

Ex. 2: Differentiation can be used to define linear transformations on function spaces. Examples:

(a) For $p(x) = a_0 + a_1 x + \dots + a_n x^n$, $p'(x) = a_1 + 2a_2 x^2 + \dots + n a_n x^{n-1}$. This determines a map $p \to T(p) = p'$ on polynomial spaces. May have $T: \mathcal{P} \to \mathcal{P}$, or $T: \mathcal{P}_n \to \mathcal{P}_n$, or $T: \mathcal{P}_n \to \mathcal{P}_{n-1}$. This is clearly linear, since $T(\alpha p + \beta q) = (\alpha p + \beta q)' = \alpha p' + \beta q' = \alpha T(p) + \beta T(q)$.

(b) Differentiation similarly can define a linear transformation on spaces of more general functions as long as the domain space has only diff'ble functions. For example, taking $C^1[0,1]$ = all continuously diff'ble functions on $[0,1]$, we have a linear transformation $T: C^1[0,1] \to C[0,1]$ defined by $T(f) = f'$.

Note: Integration can be similarly used to define linear transformations on function spaces.

In addition to naturally defining linear transformations on $\mathbb{R}^n$, matrices can be associated with general linear transformations, as follows: Suppose that $T: \mathcal{V} \to \mathcal{W}$ is linear and that $\dim \mathcal{V} = n$ and $\dim \mathcal{W} = m$. Let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ be bases of $\mathcal{V}$ and $\mathcal{W}$, resp. For $j = 1, \dots, n$, have

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

for some a_{ij}. Then

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{mn}$$

is the matrix representation of T with respect to $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$.

Note that the representation depends on the bases in $\mathcal{V}$ and $\mathcal{W}$. Different bases result in different representations. Later, will study how different representations of a transformation are related and determine some properties that are invariant among the representations and thus are properties of the transformation itself.

To continue: For $v \in \mathcal{V}$, can write $v = \sum_{j=1}^{n} \alpha_j v_j$ and $T(v) = \sum_{i=1}^{m} \beta_i w_i$.

Terminology: $\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix} \in \mathbb{R}^n$ and $\begin{pmatrix}\beta_1\\ \vdots\\ \beta_m\end{pmatrix} \in \mathbb{R}^m$ are the coordinate vectors of v with respect to $\{v_1, \dots, v_n\}$ and of w with respect to $\{w_1, \dots, w_m\}$, resp.

We have

$$T(v) = T\left(\sum_j \alpha_j v_j\right) = \sum_j \alpha_j T(v_j) = \sum_j \alpha_j \left(\sum_i a_{ij} w_i\right)$$
$$= \sum_i \left(\sum_j a_{ij}\alpha_j\right) w_i$$

Since $T(v) = \sum_i \beta_i w_i$, we must have $\beta_i = \sum_j a_{ij}\alpha_j$, i.e.,

$$\begin{pmatrix}\beta_1\\ \vdots\\ \beta_m\end{pmatrix} = A \begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}.$$

Thus A maps the coordinate vector of $v \in \mathcal{V}$ (with respect to $\{v_1, \dots, v_n\}$) to the coordinate vector of $T(v) \in \mathcal{W}$ (with respect to $\{w_1, \dots, w_m\}$).

Ex.: T = differentiation mapping $\mathcal{P}_3$ to $\mathcal{P}_2$.
Use the natural bases $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$.
Want $A \in \mathbb{R}^{3\times 4}$. Have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$1 \to 0$, $x \to 1$, $x^2 \to 2x$, $x^3 \to 3x^2$

Note: If we considered T to be a map from $\mathcal{P}_n$ to $\mathcal{P}_n$, then would have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Ex.: $A \in \mathbb{R}^{m\times n}$ naturally defines $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(x) = Ax \in \mathbb{R}^m$ for $x \in \mathbb{R}^n$. Suppose we have the natural bases $\{e_1, -, e_n\}$ on $\mathbb{R}^n$ and $\{e_1, -, e_m\}$ on $\mathbb{R}^m$. Then for $e_j \in \mathbb{R}^n$,

$$T(e_j) = Ae_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} = \sum_{i}^{m} a_{ij} e_i \in \mathbb{R}^m,$$

so T has representation A with respect to the natural bases.

Range and null space.

Two very important subspaces associated with $T: \mathcal{V} \to \mathcal{W}$:

- The range of T, $\mathcal{R}(T) = \{ \omega \in \mathcal{W} : \omega = T(v) \text{ for some } v \in \mathcal{V} \}$

- the null space of T, $\mathcal{N}(T) = \{ v \in \mathcal{W} : T(v) = 0 \}$

Terminology: Say T is onto if $\mathcal{R}(T) = \mathcal{W}$. Say T is one-to-one if for every $\omega \in \mathcal{R}(T)$, there is a unique $v \in \mathcal{V}$ such that $T(v) = \omega$. Say (sometimes) T is an isomorphism from $\mathcal{V}$ to $\mathcal{W}$ if T is both one-to-one and onto.

Suppose $\dim \mathcal{V} = n$, $\dim \mathcal{W} = m$.

Prop. 3.1. If $\{v_1, -, v_n\}$ is a basis for $\mathcal{V}$, then $\{T(v_1), \ldots, T(v_n)\}$ is a spanning set for $\mathcal{R}(T)$.

Pf: If $\omega \in \mathcal{R}(T)$, then $\omega = T(v)$ for some v. Writing $v = \sum_{i=1}^{n} \alpha_i v_i$, we have $\omega = T(\sum_i \alpha_i v_i) = \sum \alpha_i T(v_i) \in \operatorname{span}\{T(v_1), \ldots, T(v_n)\}$.