Fixed-point problem

Given $G: \mathbb{R}^n \to \mathbb{R}^n$ find x_* such that $x_* = G(x_*)$

Note: $x_* = G(x_*) \iff F(x_*) = 0$, $F(x) = x - G(x)$

Fixed point iteration*:

Given x
until termination do
$x \leftarrow G(x)$

* a.k.a. functional iteration, sometimes Picard iteration
successive substitution

Questions: When does x_* exist?
When is x_* unique?
When does fixed point iteration converge?
(globally? locally?)

How fast does it converge?

- Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^n$, inducing $\|\cdot\|$ on $\mathbb{R}^{n \times n}$
- Define G is a contraction mapping on $D \subseteq \mathbb{R}^n$ if $\exists \lambda$, $0 \le \lambda < 1$ such that $\|G(x) - G(y)\| \le \lambda \|x - y\|$ for $x, y \in D$
- see: G a contraction on $D \iff G$ Lipschitz on D with constant $\lambda < 1$
- Also, G is a contraction on D if G is continuously differentiable on D with $\|G'(x)\| \le \lambda < 1$ for $x \in D$

Why?

$$G(y) - G(x) = \int_0^1 \frac{d}{dt} G(x_t)\, dt \quad , \quad x_t = x + t(y - x)$$

$$= \int_0^1 G'(x_t)(y - x)\, dt$$

$$\Rightarrow \|G(y)-G(x)\| \le \left\{ \int_0^1 \|G'(x_t)\|\,dt \right\} \|y-x\| \le \lambda \|y-x\|$$

Theorem 1: Suppose G is a contraction mapping on closed $D \in \mathbb{R}^n$. Then $\exists$ unique fixed-point $x_* \in D$ and for any $x_0 \in D$, the fixed point iterates $\{x_k\}$ converge to x_* with

$$\|x_{k+1} - x_*\| \le \lambda \|x_k - x_*\|$$

proof:

see: $\|x_{k+1} - x_k\| = \|G(x_k) - G(x_{k-1})\| \le \lambda \|x_k - x_{k-1}\|$

$$\le \lambda^2 \|x_{k-1} - x_{k-2}\|$$

$$\le \dots$$

$$\Rightarrow \|x_{k+1} - x_k\| \le \lambda^{k-1} \|x_1 - x_0\|$$

$$\Rightarrow \|x_{k+e} - x_k\| \le \|x_{k+e} - x_{k+e-1}\| + \|x_{k+e-1} - x_{k+e-2}\| + \dots + \|x_{k+1} - x_k\|$$

$$\le \left\{ \lambda^{e-1} + \lambda^{e-2} + \dots + \lambda + 1 \right\} \|x_{k+1} - x_k\|$$

$$\le \frac{1}{1-\lambda} \|x_{k+1} - x_k\|$$

$$\le \frac{\lambda^{k-1}}{1-\lambda} \|x_1 - x_0\|$$

- So by taking k sufficiently large, can make $\|x_{k+e} - x_k\|$ as small as desired for <u>every</u> $e > 0$
  i.e. $\{x_k\}$ is <u>Cauchy</u>
- Since D is closed (and $\mathbb{R}^n$ is complete*) $\exists x_* \in D$ such that $x_k \to x_*$

  * complete: every Cauchy sequence is convergent

see: $G(x_*) = \lim_{k\to\infty} G(x_k) = \lim_{k\to\infty} x_{k+1} = x_*$

Also, $\|x_{k+1} - x_*\| = \|G(x_k) - G(x_*)\| \leq \lambda \|x_k - x_*\|$

Finally, if $\tilde{x}_* \in D$ is a fixed point then

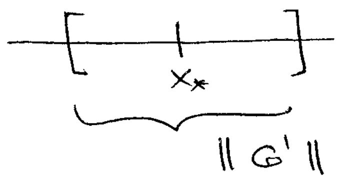
$$\|\tilde{x}_* - x_*\| = \|G(\tilde{x}_*) - G(x_*)\| \leq \lambda \|\tilde{x}_* - x_*\|$$

$$\Rightarrow \quad \tilde{x}_* - x_* = 0 \quad \text{since} \quad \lambda < 1$$

Theorem 2: Suppose G is continuously differentiable near x_* with $\|G'(x_*)\| < 1$. Then for any η with $\|G'(x_*)\| < \eta < 1$, $\exists \delta > 0$ such that if $\|x_0 - x_*\| \leq \delta$, then fixed point iterates $\{x_k\}$ converge to x_* with

$$\|x_{k+1} - x_*\| \leq \eta \|x_k - x_*\|$$

proof:



choose δ such that $\|G'(x)\| \leq \eta$ whenever $\|x - x_*\| \leq \delta$. Set $H_\delta = \{x : \|x - x_*\| \leq \delta\}$

see: H_δ is closed and

i) $x, y \in H_\delta \Rightarrow \|G(x) - G(y)\| = \left\|\left\{\int_0^1 G'(x_t)\,dt\right\}(x-y)\right\|$

$$\leq \eta \|x - y\|$$

$$\# \; x_t = y + t(x-y) \; \#$$

ii) in particular $x \in H_\delta$

$$\Rightarrow \|G(x) - x_*\| = \|G(x) - G(x_*)\|$$

$$\leq \eta \|x - x_*\| \leq \delta$$

Dissatisfying: Convergence on $\mathbb{R}^n$ is norm independent.

i.e., since $\mathbb{R}^n$ is finite-dimensional, all norms are <u>equivalent</u>**, so convergence in any norm $\Rightarrow$ convergence in every other norm

** if $\|\cdot\|$, $|||\cdot|||$ are norms on $\mathbb{R}^n$, then

$$\alpha_1 \|v\| \leq |||v||| \leq \alpha_2 \|v\|$$

α_1, α_2 fixed for any vector v

However, our sufficient condition $\|G'(x_*)\|$ is norm-dependent

For $M \in \mathbb{R}^{n \times n}$, denote

$$\sigma(M) = \{\lambda : Mx = \lambda x \text{ for some } x \neq 0\} \quad \text{(spectrum of } M)$$

$$\rho(M) = \max_{\lambda \in \sigma(M)} |\lambda| \quad \text{(spectral radius of } M)$$

Note: For any induced $\|\cdot\|$ on $\mathbb{R}^{n \times n}$ have $\rho(M) \leq \|M\|$

Why? For $\lambda \in \sigma(M)$, $Mx = \lambda x$ for some $x \neq 0$

$$\Rightarrow \|Mx\| = |\lambda| \|x\|$$

$$\Rightarrow |\lambda| = \frac{\|Mx\|}{\|x\|} \leq \|M\|$$

sometimes $\|M\| = \rho(M)$ e.g. if $M = M^T$ and $\|\cdot\| = \|\cdot\|_2$

$$M = U \Lambda U^T \quad \text{where} \quad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{and } U^T U = I$$

and can show

$$\|M\| = \|\Lambda\| = \max |\lambda_i| = \rho(M)$$

Linear algebra fact: Given any $M \in \mathbb{R}^{n \times n}$ and $\epsilon > 0$

$\exists$ induced $\|\cdot\|$ such that

$$\|M\| \leq \rho(M) + \epsilon$$

Theorem 3: Suppose x_* such that $x_* = G(x_*)$ and G is continuously differentiable near x_* with $\rho(G'(x_*)) \equiv \rho_* < 1$. Then for any η, $\rho_* < \eta < 1$, there is a $\delta > 0$ and a norm $\|\cdot\|$ such that if $\|x_0 - x_*\| \le \delta$ then fixed point iterates $\{x_k\}$ converge to x_* with

$$\| x_{k+1} - x_* \| \le \eta \| x_k - x_* \| \quad , \text{ for each } k$$

Note: if $\rho(G'(x_*)) \geq 1$ can't expect $x_k \to x_*$

$$(***) \quad ---- \quad x_{k+1} - x_* = G(x_k) - G(x_*) = G'(x_*)(x_k - x_*) + o(x_k - x_*)$$

$G(x) = Ax - b$

$G'(x) = A$

$Ax = \lambda x$, λ eigenvalue

$$x_{k+1} - x_* = A x_{k+1} - b - A x_* + b$$
$$= A(x_{k+1} - x_*) = A^{k+1}(x_1 - x_0) = \lambda^{k+1}(x_1 - x_0)$$

if $\lambda > 1 \Rightarrow x_{k+1} - x_* \not\to 0$

special case $n = 1$ Have, assuming $x_k \neq x_*$ for every k

$$G'(x_*) \leftarrow \frac{x_{k+1} - x_*}{x_k - x_*} = G'(x_*) + \frac{o(x_k - x_*)}{x_k - x_*}$$

general n: if $x_k \to x_*$ and $G'(x_*) = 0$ then

$(***) \Rightarrow$ superlinear convergence

if $\| G'(x) - G'(x_*) \| \le \gamma \| x - x_* \|$ then $G'(x_*) = 0$

$\Rightarrow$ quadratic convergence

why? $\| o(x_k - x_*) \| = \| G(x_k) - G(x_*) - G'(x_*)(x_k - x_*) \|$

$$= \left\| \int_0^1 [G'(x_t) - G'(x_*)]\,dt\,(x_k - x_*) \right\|$$

$x_* + t(x_k - x_*)$

$$\leq \frac{\gamma}{2} \| x_k - x_* \|^2$$

for $(***)$ + $G'(x_*) = 0 \Rightarrow \| x_{k+1} - x_* \| \leq \frac{\gamma}{2} \| x_k - x_* \|^2$

Special case $n=1$ again:

- If $G'(x_*) = 0$ and G is twice continuously differentiable near x_*, then

$$x_{k+1} - x_* = G(x_k) - G(x_*) =$$

$$= G'(x_*)(x_k - x_*) + \frac{G''(x_*)}{2}(x_k - x_*)^2 + o((x_k - x_*)^2)$$

$$\Rightarrow \frac{x_{k+1} - x_*}{(x_k - x_*)^2} = \frac{G''(x_*)}{2} + \frac{o((x_k - x_*)^2)}{(x_k - x_*)^2} \longrightarrow \frac{G''(x_k)}{2}$$

Application 1. Suppose have "Newton-like" iteration

$$x_{k+1} = x_k - B(x_k)^{-1} F(x_k)$$

Treat as $x_{k+1} = G(x_k)$

$$G(x) \equiv x - B(x)^{-1} F(x)$$

If B differentiable, then have

$$G'(x) = I - B(x)^{-1} F'(x) + o(x - x_*)$$

$$\Rightarrow G'(x_*) = I - B(x_*)^{-1} F'(x_*)$$

so have local linear convergence if

$$\| I - B(x_*)^{-1} F'(x_*) \| < 1$$

(special case of earlier result)

In particular $B(x) = F'(x)$ then

$$G'(x_*) = I - F'(x_*)^{-1} F(x_*) = 0$$

and set superlinear convergence (if F is twice continuously differentiable, convergence is quadratic)

see: stronger assumption to get quadratic convergence instead of Lipschitz continuity.

Application 2: Consider <u>backward differentiation formula</u> method for ODE IVP

$$y' = f(t, y)$$
$$y(t_0) = y_0$$

at m^{th}-time step of BDF of order q have

$y_{m-1}, \dots y_{m-q}$ (where $y_i \approx y(t_i)$) and want

$y_m \approx y(t_m)$ such that

$$y_m = h\beta_0 f(t_m, y_m) + a_m$$

$$a_m = \sum_{j=1}^{q} \alpha_j y_{m-j}$$

Obvious fixed point iteration:

start with $y_m^{(0)}$ given by explicit "predictor" method

Iterate:

$$y_m^{(k+1)} = G(y_m^{(k)}) \quad , \quad G(y) = h\beta_0 f(t_m, y) + a_m$$

here β_0 and $\alpha_1, \dots, \alpha_q$ are (fixed, known) method constraints

$$G'(y) = h\beta_0 \frac{\partial f}{\partial y}(t_m, y)$$

$$\|G'(y)\| < 1 \quad \text{if} \quad h < \frac{1}{\|\beta_0 \frac{\partial f}{\partial y}\|}$$

Application 3: Picard iteration for existance and uniqueness of solution of

$$\left.\begin{aligned} y' &= f(t,y) \\ y(0) &= V \end{aligned}\right\} \quad (*)$$

see: Solution $y(t)$ satisfies

$$y(t) = V + \int_0^t f(\tau, y(\tau))\, d\tau \qquad (**)$$

(and conversely)

Suppose want solution $[0,T]$

Picard iteration

given y_0, continuous on $[0,T]$

until termination

$$y_{k+1}(t) = V + \int_0^t f(\tau, y_k(\tau))\, d\tau$$

- Can we extend fixed-point theory to say anything about convergence of $\{y_k\}$?
- Assume $f(t,y)$ is continuous in (t,y) and

$$|f(t,y) - f(t,\tilde{y})| \le |y - \tilde{y}| \qquad \left(\begin{array}{c}\text{Lipschitz} \\ \text{continuous}\end{array}\right)$$

where $|\cdot|$ is a norm on $\mathbb{R}^n$

see: Since y_0 is continuous on $[0,T]$ so is each y_k

Set: $C[0,T] = \{ y : y \text{ defined and continuous on } [0,T] \}$

Then $G(y)$ defined by $G(y)(t) = V + \int_0^t f(\tau, y(\tau))\, d\tau$

maps $G : C[0,T] \to C[0,T]$

And y solves $(*)$ and $(**)$ $\iff$ $y = G(y)$

i.e. $y(t) = G(y)(t)$ for $0 \le t \le T$

Note: Our contraction mapping theorem 1 still holds if D is a Banach space* (even infinite dimensional)

* vector space with norm $\|.\|$ which is complete i.e. if $\{x_k\}$ is Cauchy in $\|.\|$, then $\exists\, x_*$ such that $x_k \to x_*$

Need suitable norm on $C[0,T]$

Choose $K > \gamma$, define

$$\|y\| = \max_{0\le t\le T} e^{-kt} |y(t)| \qquad \text{(max norm)}$$

can show: $\{y_k\}$ Cauchy in $\|.\| \Rightarrow \{y_k\}$ converges point wise uniformly on $[0,T]$ to $y_k \in C[0,T]$

$\Rightarrow \|y_k - y_*\| \to 0$

That is: $C[0,T]$ is complete in $\|.\|$

For y and z in $C[0,T]$ have

$$|G(y)(t) - G(z)(t)| = \left|V + \int_0^t f(\tau, y(\tau))\,d \; - V - \int_0^t f(\tau, z(\tau))\,d\tau\right|$$

$$= \left|\int_0^t [f(\tau,y(\tau)) - f(\tau, z(\tau))]\,d\tau\right|$$

$$\le \int_0^t |f(\tau,y(\tau)) - f(\tau, z(\tau))|\,d\tau$$

$$\le \int_0^z \gamma\, e^{k\tau} \cdot e^{-k\tau} |y(\tau) - z(\tau)|\,d\tau$$

$$\le \left\{\max_{0\le\tau\le t} e^{-k\tau} |y(\tau) - z(\tau)|\right\} \int_0^t \gamma e^{k\tau}\,d\tau$$

$$\le \|y - z\| \frac{\gamma}{K}(e^{kt} - 1)$$

So

$$e^{-kt} \left| G(y)(t) - G(z)(t) \right| \leq \frac{\gamma}{K} \left(1 - e^{-kt}\right) \| y - z \|$$

$$\leq \frac{\gamma}{K} \| y - z \|$$

$$\| G(y) - G(z) \| = \max_{0 \leq t \leq T} e^{-kt} \left| G(y)(t) - G(z)(t) \right|$$

$$\leq \frac{\gamma}{K} \| y - z \|$$

since $\frac{\gamma}{K} < 1$, $G : C[0,T] \to C[0,T]$ is a <u>contraction mapping</u>

By Th. 1. G has a unique fixed point $y \in C[0,1]$

i.e. $(*)$ and $(**)$ have unique solution y