Lipschitz Continuity

The purpose of this note is to summarize the discussion of Lipschitz continuity in class and, in particular, to make clear the distinctions between the notion of *Lipschitz continuity of a function at a point* (defined in class) and the notion of a *Lipschitz function* (defined in the text).

The following is the definition given in class of Lipschitz continuity of a function at a point.

DEFINITION 1. *A function f from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is* Lipschitz continuous at $x \in S$ *if there is a constant C such that*

$$\|f(y) - f(x)\| \leq C\|y - x\| \tag{1}$$

for all $y \in S$ sufficiently near x.

Note that Lipschitz continuity *at a point* depends only on the behavior of the function *near* that point. For f to be Lipschitz continuous at x, an inequality (1) must hold for all y sufficiently near x, but it is not necessary that (1) hold if y is not near x. Also, f may be Lipschitz continuous at other points, but different values of C may be required for (1) to hold near those points. For example, we saw in class that $f(x) = 1/x$ for $x > 0$ is Lipschitz continuous at each $x > 0$, but there is no single C for which (1) holds for all $x > 0$.

We saw in class that if f is Lipschitz continuous at x, then it is continuous at x. We also saw that if f is a real-valued function defined on $S \subset \mathbb{R}$ that is differentiable at $x \in S$, then f is Lipschitz continuous at x. In fact, this is more generally true: A function f from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is Lipschitz continuous at $x \in S$ if it is differentiable at x. This is not important to us at this time, so we omit the proof. Summarizing, we have

differentiable at $x \Rightarrow$ Lipchitz continuous at $x \Rightarrow$ continuous at x.

We also showed that the converse implications don't hold. Specifically, we saw that $f(x) = \sqrt{|x|}$ is continuous at $x = 0$ but not Lipschitz continuous there because its derivative is unbounded as x approaches zero. We also saw that $f(x) = |x|$ is Lipschitz continous at $x = 0$ but not differentiable there. In summary,

differentiable at $x \not\Leftarrow$ Lipchitz continuous at $x \not\Leftarrow$ continuous at x.

The following is Definition 5.1.5 on page 70 of the text.

Definition 2. *A function f from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is called a* Lipschitz function *if there is a constant C such that*

$$\|f(y) - f(x)\| \leq C\|y - x\| \tag{2}$$

for all x, $y \in S$.

Note that for f to be a Lipschitz function, the constant C must be such that (2) holds for *all x and y in S.* In contrast, in Definition 1, the constant C must only be such that (1) holds for all y in S sufficiently near the particular point x; as previously noted, a different C may be required for a different $x \in S$.

The following proposition (see Exercise 5.1.J in the text) may be useful in determining whether a real-valued function on an interval $I \subseteq \mathbb{R}$ is a Lipschitz function. The proof relies on the Mean Value Theorem, which isn't covered until §6.2 of the text but which should be familiar from calculus. The proposition can be extended to apply to a function f from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$, but the extension is a bit more complicated (and relies on the Fundamental Theorem of Calculus, since there is no Mean Value Theorem in higher dimensions).

Proposition 3. *Suppose that f is a real-valued function defined and differentiable on an interval $I \subset \mathbb{R}$. If f' is bounded on I, then f is a Lipschitz function on I.*

Proof. Suppose that M is such that $|f'(x)| \leq M$ for all $x \in I$. Then for x and y in I, we have (see the text, 6.2.2 Mean Value Theorem, page 99)

$$f(y) - f(x) = f'(c)(y - x)$$

for some c between x and y, and it follows that

$$|f(y) - f(x)| \leq M|y - x|.$$

Thus (2) holds with $C = M$. □

We will see in Section 5.4 of the text that a continuous function on a compact set has maximum and minimum values on that set. Thus a particular consequence of Proposition 3 is that if a real-valued function f is continuously differentiable on a *closed* interval $I \subset \mathbb{R}$, then f is a Lipschitz function on I.

Example 4. *The function $f(x) = \tan x$ is differentiable at each $x \in (-\pi/2, \pi/2)$ and, therefore, Lipschitz continuous at each $x \in (-\pi/2, \pi/2)$. However, because f' is unbounded on $(-\pi/2, \pi/2)$, there is no constant C such that* (2) *holds for all x and y in $(-\pi/2, \pi/2)$; thus f is not a Lipschitz function on $(-\pi/2, \pi/2)$. But if a and b are such that $-\pi/2 < a < b < \pi/2$, then f is continuously differentiable on $[a, b]$, and it follows that f is a Lipschitz function on $[a, b]$.*