

Derivation of the Natural Cubic Spline

Suppose we have $a = x_0 < \dots < x_n = b$ and $y_0, \dots, y_n$. A cubic interpolating spline for these data is a function $S(x)$ that is twice continuously differentiable on $[a, b]$, satisfies $S(x_i) = y_i$ for $i = 0, \dots, n$, and is such that $S(x) = S_i(x)$, a cubic polynomial, for x in $[x_i, x_{i+1}]$ and $i = 0, \dots, n - 1$. The following derivation shows how to construct the cubic “pieces” $S_i(x)$, $i = 0, \dots, n - 1$, using the “natural” end conditions $S''(x_0) = S''(x_n) = 0$.

Notation: $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_i = y_{i+1} - y_i$ for $i = 0, \dots, n - 1$.

Suppose

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \quad i = 0, \dots, n - 1.$$

We have two *interpolation conditions* on $S_i(x)$: $S_i(x_i) = S(x_i) = y_i$ and $S_i(x_{i+1}) = S(x_{i+1}) = y_{i+1}$. The first immediately gives $a_i = y_i$. With this, the second gives

$$y_i + b_i \Delta x_i + c_i \Delta x_i^2 + d_i \Delta x_i^3 = y_{i+1}.$$

Subtracting y_i from both sides and dividing by Δx_i , we obtain

$$b_i + c_i \Delta x_i + d_i \Delta x_i^2 = \frac{\Delta y_i}{\Delta x_i}, \quad i = 0, \dots, n - 1. \quad (1)$$

The *differentiability conditions* are that $S'(x)$ and $S''(x)$ exist and are continuous throughout $[a, b]$. These conditions require that $S'_i(x_{i+1}) = S'_{i+1}(x_{i+1})$ and $S''_i(x_{i+1}) = S''_{i+1}(x_{i+1})$ for $i = 0, \dots, n - 2$. With $S'_i(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2$ for each i , the first requirement becomes

$$b_i + 2c_i \Delta x_i + 3d_i \Delta x_i^2 = b_{i+1}, \quad i = 0, \dots, n - 2. \quad (2)$$

With $S''_i(x) = 2c_i + 6d_i(x - x_i)$ for each i , the second requirement becomes

$$2c_i + 6d_i \Delta x_i = 2c_{i+1}, \text{ or just } c_i + 3d_i \Delta x_i = c_{i+1}, \quad i = 0, \dots, n - 2.$$

Since we also have $S''(x_n) = S''_{n-1}(x_n) = 2c_{n-1} + 6d_{n-1}\Delta x_{n-1}$, we can *define* $c_n = S''(x_n)/2$ and extend the above relationship to $i = n - 1$, obtaining

$$c_i + 3d_i \Delta x_i = c_{i+1}, \quad i = 0, \dots, n - 1. \quad (3)$$

Note that the *natural end conditions* $S''(x_0) = S''(x_n) = 0$ imply that $c_0 = c_n = 0$. Thus, we have n b_i 's, $n - 1$ c_i 's, and n d_i 's, for a total of $3n - 1$ coefficients remaining to be determined. Equations (1), (2), and (3) are $3n - 1$ linear equations that these coefficients must satisfy.

We will manipulate (1), (2), and (3) to get a system of linear equations in the c_i 's alone. From (3), we obtain

$$d_i = \frac{c_{i+1} - c_i}{3\Delta x_i}, \quad i = 0, \dots, n-1. \quad (4)$$

Substituting this expression for d_i in (1) and (2) yields

$$b_i + c_i\Delta x_i + \frac{\Delta x_i}{3}(c_{i+1} - c_i) = \frac{\Delta y_i}{\Delta x_i} \quad i = 0, \dots, n-1, \quad (5)$$

$$b_i + 2c_i\Delta x_i + \Delta x_i(c_{i+1} - c_i) = b_{i+1} \quad i = 0, \dots, n-2. \quad (6)$$

Subtracting (5) from (6) and rearranging, we get

$$b_{i+1} = \frac{\Delta y_i}{\Delta x_i} + c_i\Delta x_i + \frac{2}{3}\Delta x_i(c_{i+1} - c_i), \quad i = 0, \dots, n-2,$$

or, replacing i by $i-1$,

$$b_i = \frac{\Delta y_{i-1}}{\Delta x_{i-1}} + c_{i-1}\Delta x_{i-1} + \frac{2}{3}\Delta x_{i-1}(c_i - c_{i-1}), \quad i = 1, \dots, n-1.$$

Substituting this expression for b_i in (5), we obtain

$$\Delta x_{i-1}c_{i-1} + 2(\Delta x_{i-1} + \Delta x_i)c_i + \Delta x_i c_{i+1} = 3\left(\frac{\Delta y_i}{\Delta x_i} - \frac{\Delta y_{i-1}}{\Delta x_{i-1}}\right), \quad i = 1, \dots, n-1. \quad (7)$$

Since $c_0 = c_n = 0$, the equations (7) constitute a system of $n-1$ equations in the $n-1$ coefficients $c_1, \dots, c_{n-1}$. Since $\Delta x_i > 0$ for each i , this system is *strictly diagonally dominant* (see Section 6.6 for the definition) and, as a consequence of Theorem 6.19, determines $c_1, \dots, c_{n-1}$ uniquely. Having solved the system (7) for the c_i 's, we can obtain the b_i 's from (5) and the d_i 's from (4).

Remark. The system (7) has another very nice property in addition to strict diagonal dominance: It is *tridiagonal*, which means that, for each i , the only non-zero coefficients in the i th equation are those of c_{i-1} , c_i , and c_{i+1} . Tridiagonal systems can be solved very economically, specifically in a small multiple of n arithmetic operations. In contrast, a general linear system, in which all coefficients may be nonzero, requires about $n^3/3$ multiplications and a similar number of additions and subtractions.