MA1022, Section B07 **MID-TERM EXAM** November 17, 2006

Use no calculators, books, or notes. *Show your work* as much as possible. Note that you only have to do two of problems 3 through 6 and two of problems 7 through 10. Look all of those problems over carefully to decide which ones to do. If there is any possibility of confusion, **clearly indicate which problems you have chosen to do**. Only those problems will be graded and counted toward your score.

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1. Evaluate the following:

a. (5 points) $\int x^4\,dx$

Solution: $\int x^4\,dx = \frac{x^5}{5} + C$.

b. (10 points) $\int \cos x\sqrt{1+\sin x}\,dx$

Solution: Set $u = 1+\sin x$. *Then* $du/dx = \cos x$, *and* $\int \cos x\sqrt{1+\sin x}\,dx = \int u^{1/2}\frac{du}{dx}\,dx = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+\sin x)^{3/2} + C$.

c. (10 points) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i}$. *Hint: Note that* $\frac{1}{n+i} = \frac{1}{1+i/n}\cdot\frac{1}{n}$.

Solution: Set $f(x) = 1/(1+x)$ *and, for a given* n, *define* $\Delta x = 1/n$ *and* $x_i = i/n$ *for* $i = 0,\ldots,n$. *Then*

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big|_0^1 = \ln 2.$$

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2. Evaluate the following:

a. (5 points) $\frac{d}{dx}\int_0^x \frac{1}{1+\sin t}\,dt$

Solution: From the Mean Value Theorem, we have that $\frac{d}{dx}\int_0^x \frac{1}{1+\sin t}\,dt = \frac{1}{1+\sin x}$.

b. (10 points) $\int_0^1 x^4\,dx$

Solution: $\int_0^1 x^4\,dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5}$.

c. (10 points) $\int_0^1 \frac{x}{1+x^2}\,dx$

Solution: Set $u = 1+x^2$. *Then* $du/dx = 2x$, *and*

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{x=0}^{x=1}\frac{1}{u}\frac{du}{dx}\,dx = \frac{1}{2}\ln u\Big|_{x=0}^{x=1} = \frac{1}{2}\ln 2.$$

Work *two* of problems 3 through 6.

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3. (10 points) Find the area between the curves $y = x^2$ and $y = x^3$ for $0 \le x \le 1$.

Solution: The area is given by

$$\int_0^1 (x^2 - x^3)\,dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

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4. (10 points) Find the volume of the solid generated by revolving about the x-axis the region bounded by the curves $y = x^2$ and $y = x^3$ for $0 \le x \le 1$.

Solution: The volume is given by

$$\int_0^1 \pi\left[(x^2)^2 - (x^3)^2\right]\,dx = \pi \int_0^1 (x^4 - x^6)\,dx$$
$$= \pi\left(\frac{x^5}{5} - \frac{x^7}{7}\right)\Big|_0^1 = \pi\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{2\pi}{35}.$$

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5. (10 points) Simpson's Rule for approximating I=$\int_a^b f(x)\,dx$ is, for even n,

$$I \approx S_n = \frac{\Delta x}{3}\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 4f(x_{n-1}) + f(x_n)\},$$

where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$ for $i = 0, \ldots, n$. If $[a,b] = [0,1]$ and $f(x) = 1 + x + x^2 + x^3$, then what is the smallest n that guarantees $|I - S_n| \le 10^{-8}$? Include in your answer the general error bound given in class and in the text.

Solution: The general error bound is

$$|I - S_n| \le \frac{M_4(b-a)^5}{180n^4} = \frac{M_4}{180n^4}$$

since $b - a = 1 - 0 = 1$ in this problem. In this bound, M_4 can be any number such for $|f^{(4)}(x)| \le M_4$ for $0 \le x \le 1$. In this problem, $f(x)$ is a cubic polynomial, so $f^{(4)}(x) = 0$ for all x. Then we can take $M_4 = 0$ and obtain $|I - S_n| = 0$ for all even n. In particular, we have $|I - S_2| = 0 \le 10^{-8}$, and so 2 (the smallest even positive integer) is the desired value of n.

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6. (10 points) A ball on the end of a spring moves up and down with velocity $v(t) = \sin t$ feet per second. Find the net distance and the total distance traveled between time $t = 0$ and time $t = 2\pi$.

Solution: The net distance is given by

$$\int_0^{2\pi} v(t)\,dt = \int_0^{2\pi} \sin t\,dt = -\cos t\Big|_0^{2\pi} = 0.$$

The total distance is given by

$$\int_0^{2\pi} |v(t)|\,dt = \int_0^{\pi} \sin t\,dt - \int_{\pi}^{2\pi} \sin t\,dt = -\cos t\Big|_0^{\pi} + \cos t\Big|_{\pi}^{2\pi} = 4.$$

Work *two* of problems 7 through 10.

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7. (15 points) At time $t = 0$, a ball is dropped from rest at an initial height of 64 feet. Assume the acceleration of gravity is -32 feet per second per second. At what time does the ball hit the ground, and what is its velocity on impact?

Solution: In general, we have that $v(t) = -32t + v_0$ and $x(t) = -16t^2 + v_0 t + x_0$, where v_0 and x_0 are, respectively, the initial position and velocity. We're given that $v_0 = 0$ and $x_0 = 64$, so $v(t) = -32t$ and $x(t) = -16t^2 + 64$. The ball hits the ground when $0 = x(t) = -16t^2 + 64$, and solving for t gives $t = 2$ seconds. The velocity on impact is then $v(2) = -64$ feet per second.

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8. (15 points) The population of the US between 1900 and 2000 is *very* roughly given by $y(t) = 75e^{k(t-1900)}$ million, where t is the date in years. Find an expression in k that gives the average population between 1900 and 2000. (Don't leave the answer in integral form.)

Solution: The average population is given by

$$\frac{\int_{1900}^{2000} 75e^{k(t-1900)}\,dt}{2000-1900} = \frac{3}{4}\int_{1900}^{2000} e^{k(t-1900)}\,dt.$$

Set $u = k(t-1900)$. Then $du/dt = k$, and the average population is given by

$$\frac{3}{4k}\int_{1900}^{2000} e^u \frac{du}{dt}\,dt = \frac{3}{4k}e^u\Big|_{t=1900}^{t=2000} = \frac{3}{4k}\left(e^{100k} - 1\right).$$

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9. (15 points) Find the length of the smooth arc $y = \frac{x^2}{4} - \frac{\ln x}{2}$ from $x = 1$ to $x = 2$. (You *can* evaluate the integral.)

Solution: We have $dy/dx = \frac{x}{2} - \frac{1}{2x}$*, and the arc length is*

$$\int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}}\, dx$$

$$= \int_1^2 \sqrt{\frac{x^2}{4} + \frac{1}{x} + \frac{1}{4x^2}}\, dx = \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx$$

$$= \int_1^2 \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \frac{x^2}{4} + \frac{\ln x}{2}\bigg|_1^2 = \frac{3}{4} + \frac{\ln 2}{2}.$$

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10. (15 points) Find the area of the surface of revolution generated by revolving $y = x^3/3$, $0 \le x \le 1$, around the x-axis. (You *can* evaluate the integral.)

Solution: We have $dy/dx = x^2$*, and the surface area is*

$$\int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \frac{2\pi}{3} \int_0^1 x^3 \sqrt{1 + x^4}\, dx.$$

Set $u = 1 + x^4$*. Then* $du/dx = 4x^3$*, and the area is*

$$\frac{\pi}{6} \int_0^1 u^{1/2} \frac{du}{dx}\, dx = \frac{\pi}{6} \cdot \frac{2}{3} u^{3/2}\bigg|_{x=0}^{x=1} = \frac{\pi}{9}(1 + x^4)^{3/2}\bigg|_0^1 = \frac{\pi}{9}(2\sqrt{2} - 1).$$
