MA 2051 Notes D20': Week 7

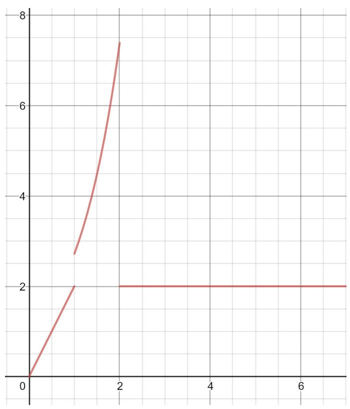
Step Functions and Delayed Functions

We let $u(t)$ denote the (unit) step function defined as $u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$

This step function can be translated to any starting point t_0 by $u(t - t_0) = \begin{cases} 0 & \text{if } t < t_0 \\ 1 & \text{if } t \geq t_0 \end{cases}$

Example: Write the given function in terms of $u(t)$ and sketch the graph. $f(t) = \begin{cases} 2t & (0 \leq t < 1) \\ e^t & (1 \leq t < 2) \\ 2 & (2 \leq t) \end{cases}$

$$f(t) = 2tu(t) + (e^t - 2t)u(t-1) + (2 - e^t)u(t-2)$$



Now suppose $\mathcal{L}\{f(t)\} = F(s)$, and c is a positive constant. We have the following two identities:

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

and

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u(t-c)f(t-c).$$

Be careful when taking the Laplace transform! This identity requires that the function f is of the form $f(t-c)$.

Example: Compute $\mathcal{L}\left\{t^2u(t-2)\right\}$.
Note that we first have to convert t^2 to a function of $(t-2)$.

$$\begin{aligned}
\mathcal{L}\left\{t^2u(t-2)\right\} &= \mathcal{L}\left\{(t^2 - 4t + 4 + 4t - 4)u(t-2)\right\} \\
&= \mathcal{L}\left\{(t^2 - 4t + 4)u(t-2) + (4t-4)u(t-2)\right\} \\
&= \mathcal{L}\left\{(t-2)^2u(t-2)\right\} + \mathcal{L}\{4(t-1)u(t-2)\} \\
&= \mathcal{L}\left\{(t-2)^2u(t-2)\right\} + \mathcal{L}\{4(t-1-1+1)u(t-2)\} \\
&= \mathcal{L}\left\{(t-2)^2u(t-2)\right\} + \mathcal{L}\{4(t-2)u(t-2) + 4(1)u(t-2)\} \\
&= \mathcal{L}\left\{(t-2)^2u(t-2)\right\} + 4\mathcal{L}\{(t-2)u(t-2)\} + 4\mathcal{L}\{u(t-2)\} \\
&= e^{-2s}\frac{2!}{s^3} + 4e^{-2s}\frac{1!}{s^2} + 4e^{-2s}\frac{1}{s}
\end{aligned}$$

Note that the constant 2 does not appear in the Laplace transform of $f(t)$, but only in the exponential e^{-2t} multiplying it.

Example: Compute $\mathcal{L}^{-1}\left\{\frac{se^{-4s}}{(3s+2)(s-2)}\right\}$.

Notice that we can rewite this function as $e^{-4s}F(s)$ where $F(s) = \frac{s}{(3s+2)(s-2)}$. Using the inveerse transform formula above, we can see that $\mathcal{L}^{-1}\left\{e^{-4s}F(s)\right\} = u(t-4)f(t-4)$. So first we need to find the inverse transform of $F(s)$:

$$\begin{aligned}
\frac{s}{(3s+2)(s-2)} &= \frac{A}{3s+2} + \frac{B}{s-2} \\
\Rightarrow s &= A(s-2) + B(3s+2) \\
s = -2/3 \qquad A &= \frac{1}{4} \\
s = 2 \qquad B &= \frac{1}{4} \\
\text{So } \mathcal{L}^{-1}\left\{\frac{s}{(3s+2)(s-2)}\right\} &= \frac{1/4}{3s+2} + \frac{1/4}{s-2} \\
&= \frac{1/4}{3}\frac{1}{s+2/3} + \frac{1}{4}\frac{1}{s-2} \\
&= \frac{1}{12}e^{-2t/3} + \frac{1}{4}e^{2t} = f(t)
\end{aligned}$$

However we are not done yet, as this is $f(t)$, so we need to plug in $t-4$:

$$f(t-4) = \frac{1}{12}e^{-2(t-4)/3} + \frac{1}{4}e^{2(t-4)}$$

All together, this gives us the inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{se^{-4s}}{(3s+2)(s-2)}\right\} = u(t-4)\frac{1}{12}e^{-2(t-4)/3} + u(t-4)\frac{1}{4}e^{2(t-4)}$$

Practice Problems

1. Find the inverse Laplace transform of the following function

   i. $\mathcal{L}\{f(t)\} = \frac{s^2-4s-12}{(s^2+4)(s-1)}$

2. Solve the initial value problem using the Laplace transform

   i. $y' - y = 4, \quad y(0) = -1$

   ii. $y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = -1$

   iii. Write the given function in terms of the unit step function and sketch the graph $f(t) = \begin{cases} t^2 & (0 \le t < 2) \\ 3 & (2 \le t < 3) \\ 2\cos(t) & (3 \le t) \end{cases}$