MA 2051 Notes D20': Week 6

Laplace Transforms

Laplace transforms can often be computed using a table of identities:

$$\mathcal{L}\{1\} = \frac{1}{s} \tag{1}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, (n = 1, 2, \dots) \tag{2}$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, s > a \tag{3}$$

$$\mathcal{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2} \tag{4}$$

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} (n = 1, 2, \dots) \tag{5}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}, s > 0 \tag{6}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}, s > 0 \tag{7}$$

Refer to the table provided on canvas for more identities.
Observe the following properties of the Laplace transform:

1. **Linearity**. Let f and g be functions whose Laplace transform exists on a common domain, and c an arbitrary constant. Then:

$$\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\} \quad \text{and} \quad \mathcal{L}\{cf\} = c\mathcal{L}\{f\}$$

2. **Derivatives**. If $f, f', f'', \cdots, f^{(n-1)}$ are continuous, $f^{(n)}$ is piecewise continuous, all these functions are of exponential order $e^{\alpha t}$, then the Laplace transfor of all this funtions exists for $s > \alpha$ and:

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0),\ \mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0),\ \cdots,\ \mathcal{L}\left\{f^{(n)}\right\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

3. **Translation Property** If the Laplace transform $F(s) = \mathcal{L}\{f\}$ exists for $s > \alpha$, then for $s > \alpha + a$:

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s - a)$$

4. **Derivatives of Transforms**. If $f(t)$ is piecewise continuous and of exponential order $e^{\alpha t}$, then for $s > \alpha$ and n a positive integer,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}(s)$$

Example. Show that the Laplace transform is linear. That is, show that for any functions f and g, and any constant c

1. $\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
2. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$

(1) $\mathcal{L}\{f+g\} = \int_0^\infty (f(t)+g(t))e^{-st}dt = \int_0^\infty f(t)e^{-st}+g(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt+\int_0^\infty g(t)e^{-st}dt = \mathcal{L}\{f\}+\mathcal{L}\{g\}$

Showing (2) is left as an exercise.

Example: Compute the Laplace transform of $5e^{3t}\sin 2t + t^2\cos 3t$

By the table of Laplace transform, we have $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$, $\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9}$

$$\begin{aligned}
\mathcal{L}\left\{5e^{3t}\sin 2t + t^2\cos 3t\right\} &= 5\mathcal{L}\left\{e^{3t}\sin 2t\right\} + \mathcal{L}\left\{t^2\cos 3t\right\} \\
&= 5\frac{2}{(s-3)^2+4} + (-1)^2\left(\frac{s}{s^2+9}\right)'' \\
&= \frac{10}{s^2-6s+13} + \frac{-2s(s^2+9)^2-(9-s^2)2(s^2+9)2s}{(s^2+9)^4} \\
&= \frac{10}{s^2-6s+13} + \frac{2s^5-36s^3-486s}{(s^2+9)^4}
\end{aligned}$$

Inverse Laplace Transforms

The *inverse Laplace Transform* can be computed by definition, or by using a table similar to that on the previous page. Note that this can often require **partial fraction decomposition** of $F(s)$.

Inverse Laplace transforms can often be computed using a table of identities:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \tag{8}$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, (n = 1, 2, \dots) \tag{9}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}, s > a \tag{10}$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin bt, s > 0 \tag{11}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos bt, s > 0 \tag{12}$$

Observe the Linearity property of the Inverse Laplace transform:

Let F and G be functions whose Inverse Laplace transform exists, and c an arbitrary constant. Then:

$$\mathcal{L}^{-1}\{F+G\} = \mathcal{L}^{-1}\{F\} + \mathcal{L}^{-1}\{G\} \quad \text{and} \quad \mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}$$

Example: Find the inverse Laplace transform of $F(s) = \frac{s+1}{(s-4)(s+2)}$

We must perform partial fractions

$$\frac{A}{s-4} + \frac{B}{s+2} = \frac{s+1}{(s-4)(s+2)} \to \frac{5/6}{s-4} + \frac{1/6}{s+2} = \frac{s+1}{(s-4)(s+2)}$$

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s-4)(s+2)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{\frac{5}{6}}{s-4}+\frac{\frac{1}{6}}{s+2}\right\} \\
&= \frac{5}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}+\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} \\
&= \frac{5}{6}e^{4t}+\frac{1}{6}e^{-2t},
\end{aligned}$$

where the inverse Laplace transforms are found by the table and linearity property.

Initial Value Problems Using Laplace Transforms

Laplace transforms and inverse Laplace transforms can be used to solve initial value problems. Consider the general problem:

$$\begin{aligned}
ay'' + by' + cy &= f(t) \\
y(0) &= y_0 \\
y'(0) &= y_0'.
\end{aligned}$$

Solving this initial value problem using Laplace transforms can be broken down into 3 simple steps.

1. First, take the Laplace transform of both sides of the equation. Using the properties of the Laplace transform, this gives:

$$a\big(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\big) + b\big(s\mathcal{L}\{y\} - y(0)\big) + c\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

   This can be simplified to:

$$(as^2 + bs + c)\mathcal{L}\{y\} = \mathcal{L}\{f(t)\} + a(sy_0 + y_0') + by_0$$

2. Next, solve this equation for $\mathcal{L}\{y\}$.

$$\mathcal{L}\{y\} = \frac{\mathcal{L}\{f(t)\} + a(sy_0 + y_0') + by_0}{(as^2 + bs + c)}$$

3. Finally, take the **inverse** Laplace transform of both sides.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{f(t)\} + a(sy_0 + y_0') + by_0}{(as^2 + bs + c)}\right\}$$

If we are able to compute this inverse Laplace transform then we have the solution to to the initial value problem. Note that taking the inverse Laplace transform of the right hand side often requires using **partial fractions.**

Example: Use the Laplace transform to solve folowing IVP:

$$y'' - 3y' + 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

$$
\begin{aligned}
\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \mathcal{L}\{0\} \\
\mathcal{L}\{LHS\} &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} \\
&= (s^2 - 3s + 2)\mathcal{L}\{y\} - s + 3 \\
\mathcal{L}\{RHS\} &= \mathcal{L}\{0\} = 0 \\
\mathcal{L}\{y\} &= \frac{s-3}{s^2 - 3s + 2} \\
\frac{s-3}{s^2-3s+2} &= \frac{s-3}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1} \\
A &= -1, B = 2 \\
y &= \mathcal{L}^{-1}\left\{\frac{s-3}{s^2-3s+2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \\
&= 2e^t - e^{2t}
\end{aligned}
$$

Practice:

1. Find the Laplace transform of the following function
   - i. $f(t) = t + 2e^{2t}$ (by definition and using table)
   - ii. $f(t) = 5t^3 + 3t^2 + 2t - 10$
   - iii. $f(t) = 2\cos(3t) + 3\sin(2t)$
2. For each function $f(t)$, find $\mathcal{L}\{f'(t)\}$ and $\mathcal{L}\{f''(t)\}$.
   - i. $f(t) = t + 2e^{2t}$
   - ii. $f(t) = 2t - 10$
   - iii. $f(t) = 2\cos(3t) + 3\sin(2t)$
3. Find the Inverse Laplace transform of the following function
   - i. $F(s) = \frac{3}{s^2}$
   - ii. $F(s) = \frac{3s}{s^2+4}$
   - iii. $F(s) = \frac{2s}{s^4+4s^2-2s^3-8s}$
4. Solve the initial value problem using the Laplace transform
   - i. $y' - y = 4, \quad y(0) = -1$
   - ii. $y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = -1$