MA 2051 Notes D'20: Week 5

Phase Angle-Amplitude Form

In this section, we consider the initial value problem

$$m\ddot{u} + ku = 0, \quad u(0) = u_0, \quad \dot{u}(0) = v_0$$

and its solution $u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$, where $c_1 = u_0$ and $c_2 = v_0/\omega_0$.

Use the following formulas to determine the **natural (angular) frequency** ω_0, the **amplitude** A, and the **phase angle** δ or ϕ.

$$\omega_0 = \sqrt{\frac{k}{m}}, \qquad A = \sqrt{c_1^2 + c_2^2} = \sqrt{u_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} \tag{1}$$

and δ (or ϕ) satisfies

$$\sin\delta = \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \cos\delta = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \Longrightarrow \quad \delta = \arctan\left(\frac{c_2}{c_1}\right) = \arctan\left(\frac{v_0}{\omega_0 u_0}\right) \tag{2}$$

$$\sin\phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \cos\phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \quad \Longrightarrow \quad \phi = \arctan\left(\frac{c_1}{c_2}\right) = \arctan\left(\frac{\omega_0 u_0}{v_0}\right) \tag{3}$$

The first two are straightforward, but the tricky part is determining δ or ϕ. You will have two choices for δ or ϕ such that $0 \le \delta, \phi \le 2\pi$ and you will need to make sure you pick the correct one so that as to satisfy the first two equations in (2) or (3).

Example: Suppose $c_1 = 1$ and $c_2 = -1$. Then $\delta = \arctan(\frac{c_2}{c_1}) = \arctan(-1) = \frac{3}{4}\pi$ or $\frac{7}{4}\pi$. Now we need to pick the right δ: note that we need an angle δ such that

$$\sin\delta = \frac{c_2}{\sqrt{c_1^2 + c_2^2}} = \frac{-1}{\sqrt{2}} < 0 \quad \text{and} \quad \cos\delta = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} = \frac{1}{\sqrt{2}} > 0$$

Thus we need to pick $\delta = \frac{7}{4}\pi$ since this is the angle that has $\sin\delta < 0$ and $\cos\delta > 0$ as required.

Once we have all of our coefficients, we use the formula:

$$u(t) = A\cos(\omega_0 t - \delta) \qquad \text{OR} \qquad u(t) = A\sin(\omega_0 t + \phi).$$

These solutions are equivalent!
Observe that from Equations (1), (2), and (3),

$$c_1 = A\cos\delta = A\sin\phi \quad \text{and} \quad c_2 = A\sin\delta = A\cos\phi \tag{4}$$

Now recall the angle sum formulas:

$$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a \quad \text{and} \quad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \tag{5}$$

Combining gives us:

$$\begin{aligned}
u(t) &= A\cos(\omega_0 t - \delta) \\
&= A\cos(\omega_0 t)\cos(\delta) + A\sin(\omega_0 t)\sin(\delta) && \text{by (5)} \\
&= (A\cos\delta)\cos(\omega_0 t) + (A\sin\delta)\sin(\omega_0 t) \\
&= c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) && \text{by (4)} \\
&= A\sin(\phi)\cos(\omega_0 t) + A\cos(\phi)\sin(\omega_0 t) && \text{by (4)} \\
&= A\sin(\omega_0 t + \phi) && \text{by (5)}
\end{aligned}$$

Thus these two solutions really are equivalent. Which one you use simply depends upon if you solve for δ or ϕ. Note that c_1 is the coefficient of cos and c_2 is the coefficeint of sin, making (2) and (3) different.

Practice Problems

1. A mass of 100g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, and if there is no damping, formulate an initial value problem for this system. Solve the initial value problem and find the amplitude, frequency, and phase angle of the motion.
2. A vertical spring stretches 4 in when a weight of 16 lbs is hung from the end. This mass is set in motion by pulling it down 1 ft and releasing it with a downward velocity of $2\sqrt{2}$ ft/s. Derive the phase angle-amplitude form of the solution.

Spring Models

We are presented with two models:

1. The first type of model has a first derivative term. This model is known as the *damped* model.

$$m\ddot{u} + d\dot{u} + ku = 0, \quad u(0) = u_0, \quad \dot{u}(0) = v_0$$

2. The second type of model has a nonhomogeneous part. This is known as the *forced* model. This model can be damped or undamped.

$$m\ddot{u} + d\dot{u} + ku = F_0 \cos \omega t, \quad u(0) = u_0, \quad \dot{u}(0) = v_0$$

When constructing models, remember that to determine the damping constant d:

a. In the case of one mass on the table and one mass dangling, $d = \frac{w_s}{v_t}$ where w_s is the weight of the mass on the spring and v_t is the velocity of the mass on the table when the weight is hung.

b. In the case of only one mass, $d = \frac{w_r}{v_m}$ where w_r is the weight of the resistance and v_m is the velocity of the mass with this given resistance.

Example A mass weighting 64 pounds stretches a spring 5.12 feet. Assuming a damping force that is 10 times the velocity of the mass, determine the equation of the motion, if the mass is initially released from a point 1 foot below the equilibrium position with a upward velocity of 5 ft/s. Determine when the object passes through the equilibrium position.

$$k = \frac{w}{\Delta L} = \frac{64lb}{5.12ft} \approx 12.5lb/ft, m = \frac{w}{g} = \frac{64lb}{32} = 2slugs, d = 10$$
$$m\ddot{u} + d\dot{u} + ku = 0 \rightarrow 2\ddot{u} + 10\dot{u} + 12.5u = 0, u(0) = 1, \dot{u} = -5$$
$$\text{Solve this differential equation, we get } x(t) = e^{-2.5t}(c_1 + c_2 t)$$
$$\text{By the initial conditions, we get } c_1 = -5, c_2 = -2.5$$
$$x(t) = e^{-2.5t}(1 - 2.5t)$$
$$0 = e^{-2.5t}(1 - 2.5t) \rightarrow t = 4.5 \text{ sec}$$

Resonance

Resonance occurs when the external frequency of the forcing function is the same as the natural vibrating frequency ω_0 of the solution. We focus on the undamped case, where the resonant frequency is *always* the natural frequency $\omega_0 = \sqrt{k/m}$.

Ex. If we have the following equation for our undamped oscillator:

$$x'' + 9x = \sin 3t$$

Then the homogeneous solution would be

$$u_h(t) = C_1 \sin 3t + C_2 \cos 3t.$$

The particular solution would be $u_p = At \sin 3t + Bt \cos 3t$, which contains a factor of t, so regardless of the values of c_1 and c_2, it will oscillate with increasing amplitude as t increases. This represent the phenomenon of resonance and the resonant frequency is $\omega = \sqrt{k/m} = \sqrt{9/1} = 3$.

Practice Problems

A mass weighing 4 lb stretches a spring 2 in. The mass is then pulled down 6 in. from equilibrium and released. The mass is in a fluid with resistance 2 lb/(ft/sec). Formulate the initial value problem for this system. Solve the initial value problem and determine the damped amplitude, damped frequency, and damped period.

Laplace Transforms

The Laplace Transform is a useful tool in solving many types of differential equations since it transforms a differential equation into an ploynomial equation.
Let $f(t)$ be a function defined on $(0, \infty)$. The **Laplace transform** of $f(t)$ is the function $F(s)$ defined by the following (improper) integral:

$$F(s) = \mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t)\, dt$$

If asked to compute the Laplace transform using the definition, we carry out this improper integral, often using integration by parts. The Laplace transform is a function of s, where the domain of $F(s)$ is taken to be all values of s for which the integral exists.

Example. Compute the Laplace transform of $f(t) = 2$ using the definition of the transform.

$$\mathcal{L}\{2\} = \int_0^\infty 2e^{-st} dt = \lim_{b\to\infty} \int_0^b 2e^{-st} dt = \lim_{b\to\infty} \frac{-2}{s} e^{-st}\Big|_0^b = \lim_{b\to\infty} (\frac{-2}{s} e^{-sb} - \frac{-2}{s}) = \frac{2}{s}$$

Practice Problems

Compute the Laplace transform of the following functions:

1. $f(t) = 1$
2. $f(t) = t^2$
3. $f(t) = 2 + 3t^2$
4. $f(t) = \cos(2t)$