MA 2051 Notes D'20: Week 4

Variation of Parameters

Variation of parameters is a method based on the assumption that our particular solution consists of variables times each linearly independent part of our homogeneous solution, i.e., given the differential equation

$$y'' + p(x)y' + q(x)y = f(x) \text{ with initial condition } y(x_i) = y_i$$

we will have particular solution

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

Like the method of undetermined coefficients, we substitute this into our differential equation and solve for $v_1(x)$ and $v_2(x)$, which will give us a particular solution $y_p(x)$. This is shown in the text, but is summarized as follows:
First consider that we want to find functions v_1 and v_2 such that $y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$. By plugging in to the differential equation, we arrive at the following linear system:

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1 + v_2'y_2 &= f(t) \end{aligned}$$

which yields solutions for v_1' and v_2'

$$v_1' = -\frac{y_2(t)f(t)}{W[y_1, y_2](t)} \qquad \text{and} \qquad v_2' = \frac{y_1(t)f(t)}{W[y_1, y_2](t)}$$

Then by simple integration, we get

$$v_1(x) = -\int \frac{y_2(t)f(t)}{W[y_1, y_2](t)}\,dt, \qquad v_2(x) = \int \frac{y_1(t)f(t)}{W[y_1, y_2](t)}\,dt.$$

Then we plug in to the formula for y_p to get the desired result. Below is a three step method to find the particular solution y_p of $y'' + p(x)y' + q(x)y = f(x)$ with initial condition $y(x_i) = y_i$.

Note: If the equation is NOT in the form $y'' + p(x)y' + q(x)y = f(x)$, you will have to put it in this form before applying the following steps!

(i) Find a nontrivial solution $y_h(x) = C_1y_1(x) + C_2y_2(x)$ of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

(ii) Then we have the formula

$$y_p(x) = -y_1(x)\int \frac{y_2(t)f(t)}{W[y_1, y_2](t)}\,dt + y_2(x)\int \frac{y_1(t)f(t)}{W[y_1, y_2](t)}\,dt$$

where $W[y_1, y_2]$ is the Wronskian $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

(iii) The general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x)$$

Now we may apply the initial condition $y(x_i) = y_i$ to solve for C_1 and C_2.
Note: It is worth mentioning that sometimes solving the system of linear equations may be more convenient (and practical) than computing the Wronskian and applying the formula for y_p.

Practice Problems: Find the general solution to the differential equation
a) $y'' - 5y' + 6y = 2e^t$, b) $y'' + y = \tan t$, $0 < t < \pi/2$, c) $y'' + 4y' + 4y = t^{-2}e^{-2t}$, $t > 0$

Spring Models

We are presented with the *undamped* model $m\ddot{u} + ku = 0$, $u(0) = u_0$, $\dot{u}(0) = v_0$
When constructing models, remember that

1. Determining the constant m (mass): $m = \frac{weight}{g}$. This is the mass of the object we are modeling. To find the mass (in slug) divide the weight (in pounds) by $g = 32\text{ft/sec}^2$.
2. Determining the constant k (spring constant): $k = \frac{weight}{displacement}$

   Note: this is WEIGHT not MASS! If mass in slug, multiply by $g = 32\text{ft/sec}^2$ to get weight.
3. The initial conditions will be explicitly stated, but sometimes they can be disguised.

   a. The initial position, unless explicity stated, may be the displacement of the mass when the weight is hung. This is u_0.

   b. The initial velocity will usually be stated, unless it is released from rest/equilibrium. This is v_0.