MA 2051 D'20 Notes: Week 3

Intro to Second-order Linear Equations

Any differential equation $y'' + p(x)y' + q(x)y = f(x)$, has an associated **homogeneous equation** obtained by simply replacing $f(x)$ with zero:

$$y'' + p(x)y' + q(x)y = 0$$

Solving nonhomogeneous differential equations can be broken down into two steps.

i. Find the *homogeneous solution*, denoted y_h

ii. Find a *particular solution*, denoted y_p.

The **principle of superposition** states that any linear combination of solutions of the homogeneous equation will solve the homogeneous equation. Moreover, if y_h solves the homogeneous equation and y_p satisfies the nonhomogeneous equation, then $y_h + y_p$ solves the nonhomogeneous equaiton $y'' + p(x)y' + q(x)y = f(x)$.

Linear independence and the Wronskian

Two functions are **linearly dependent** on an interval $I = (a, b)$ if and only if there exists constants k_1, k_2, not both zero, such that:

$$k_1 f(x) + k_2 g(x) = 0, \text{ for all } x \in I$$

Otherwise, they are called **linearly independent**. The **Wronskian** is a useful tool for determining linear independence. Given functions y_1, y_2, the Wronskian is defined as:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

The two functions are *linearly independent* on an interval $I = (a, b)$ if the Wronskian is not the zero function on (a, b): i.e. there exists some point x_0 in (a, b) such that $W(y_1, y_2)(x_0) \neq 0$.

Example: On the interval $(0, 2\pi)$ are the functions $y_1 = \sin(x)$ and $y_2 = \sin(2x)$ linearly independent? Here, $y_1' = \cos(x)$ and $y_2' = 2\cos(2x)$. So $W(y_1, y_2) = 2\sin(x)\cos(2x) - \sin(2x)\cos(x)$. Consider the point $x_0 = \pi/2$. Then $W(y_1, y_2)(\pi/2) = 2\sin(\pi/2)\cos(\pi) - \sin(\pi)\cos(\pi/2) = -2 \neq 0$ Therefore $W(y_1, y_2)$ is not zero at all points of $(0, 2\pi)$, so y_1 and y_2 are linearly independent on $(0, 2\pi)$.

Undetermined Coefficients

The method of undetermined coefficients is used to find a particular solution $y_p(x)$ of the linear *nonhomogeneous* equation

$$ay'' + by' + cy = f(x)$$

when the nonhomogeneous term $f(x)$ is either $x^n, e^{ax}, \sin bx, \cos bx$ or products of these functions. The method proceeds as follows:

1. Find solution y_h of the corresponding homogeneous equation $ay'' + by' + cy = 0$
2. Make a guess for particular solution y_p based on the nonhomogeneous term $f(x)$
3. Compare guess y_p and homogeneous solution y_h. If *any* term of y_h matches any term of your guess for y_p, multiply y_p by x^s ($s = 0, 1$ or 2). Choose s to be the lowest value that will give you no conflicting terms in y_h and y_p.

$f(x)$	$y_p(x)$
a (constant)	$x^s \cdot A$ (constant)
$a_0 + a_1x + \ldots + a_nx^n$	$x^s \cdot (A_0 + A_1x + \ldots + A_nx^n)$
ae^{ax}	$x^s \cdot (Ae^{ax})$
$a\cos kx + b\sin kx$	$x^s \cdot (A\cos kx + B\sin kx)$
$(a_0 + a_1x + \ldots + a_nx^n)e^{qx}$	$x^s \cdot (A_0 + A_1x + \ldots + A_nx^n)e^{qx}$
$(a_0 + a_1x + \ldots + a_nx^n)\cos px + (b_0 + b_1x + \ldots + b_nx^n)\sin px$	$x^s \cdot \Big((A_0 + A_1x + \ldots + A_nx^n)\cos px + (B_0 + B_1x + \ldots + B_nx^n)\sin px\Big)$
$e^{ax}\Big((a_0 + a_1x + \ldots + a_nx^n)\cos px + (b_0 + b_1x + \ldots + b_nx^n)\sin px\Big)$	$x^s \cdot \Big((A_0 + A_1x + \ldots + A_nx^n)\cos px + (B_0 + B_1x + \ldots + B_nx^n)\sin px\Big)e^{ax}$

Example: Find the general solution of $y'' + y' + y = x^2$.
Step 1. Find the homogeneous solution $y_h(x)$.
Here $a = 1, b = 1, c = 1$, we have $b^2 - 4ac = -3 < 0$ and $y_h(x) = e^{-x/2}(c_1\cos(\frac{\sqrt{3}}{2}x) + c_2\sin(\frac{\sqrt{3}}{2}x))$.
Step 2. Find the particular solution $y_p(x)$.
Here the right hand side of the ODE is x^2, we set the form of $y_p(x) = x^s \cdot (A_0 + A_1x + \ldots + A_nx^n)$. Since the highest order of x^2 is two, we have $n = 2$. Since there is no similar polynomial form in the homogeneous solution y_h, we have $s = 0$. So $y_p(x)$ has the form $y_p(x) = A_0 + A_1x + A_2x^2$.
To plug $y_p(x)$ into the original equation, we compute $y_p'(x) = A_1 + 2A_2x$, $y_p'' = 2A_2$. Plugging all $y_p(x), y_p'(x), y_p''$ into the original ODE, $2A_2 + (A_1 + 2A_2x) + (A_0 + A_1x + A_2x^2) = x^2$. By solving this equations (similar with methods of Partial Fraction), we have $A_2 = 1, 2A_2 + A_1 = 0, 2A_2 + A_1 + A_0 = 0$ and $A_0 = 0, A_1 = -2, A_2 = 1$. So $y_p(x) = x^2 - 2x$.
Step 3. Find the general solution. $y(x) = y_h(x) + y_p(x) = e^{-x/2}(c_1\cos(\frac{\sqrt{3}}{2}x) + c_2\sin(\frac{\sqrt{3}}{2}x)) + x^2 - 2x$.

Note: This method can also be used if the nonhomogeneous term is a sum of the forms described in the previous rules. In this case in order to determine a guess for the particular solution we will need to use the principle of superposition. For a reference see Example 9 p.155 of the textbook.

Practice Problems:

1. Find the general solution using the method of undetermined coefficients
   i. $y'' + 2y' + 5y = 3\sin 2t$ ii. $y'' + 4y' + 4y = 2e^{-2t}$
2. Solve the initial value problem $y'' + y' - 2y = 2t$, $y(0) = 0$, $y'(0) = 1$
3. Guess the form of $y_p(x)$ if the method of undet. coeffs is to be used. DO NOT solve for the coefficients.
   i. $y'' - 5y' + 6y = e^t\cos 2t + e^{2t}(3t + 4)\sin t$ ii. $y'' + y = 2x^4 + x^2e^{-3x} + \sin 3x$ iii. $y'' + y = t(1 + \sin t)$

More On Undetermined Coefficients

Undetermined coefficients here is the same thing, except we need to be a little more cautious about multiplying by x, x^2, etc. Since we have more options for homogeneous solutions we will have more cases where this extra x or x^2 is necessary. Follow the three steps to be safe:

1. Find homogeneous solution y_h
2. Make guess for particular solution y_p based *only* on forcing function (right hand side)
3. Compare guess for y_p and homogeneous solution y_h. If *any* part of y_h conflicts with your guess for y_p, multiply conflicting term (or terms) by x^s where s is the lowest number that will give you no conflicting (linearly dependent) terms in your homogeneous and particular solutions.

Forcing Term $f(x)$	Proposed Particular Solution
$a_0 + a_1 x + \ldots + a_n x^n$	$A_0 + A_1 x + \ldots + A_n x^n$
$(a_0 + a_1 x + \ldots + a_n x^n)e^{qx}$	$(A_0 + A_1 x + \ldots + A_n x^n)e^{qx}$
$(a_0 + a_1 x + \ldots + a_n x^n)\cos px$	$(A_0 + A_1 x + \ldots + A_n x^n)\cos px$
$+(b_0 + b_1 x + \ldots + b_n x^n)\sin px$	$+(B_0 + B_1 x + \ldots + B_n x^n)\sin px$
$ae^{qx}\cos px + be^{qx}\sin px$	$Ae^{qx}\cos px + Be^{qx}\sin px$

Practice Problems:

3. Find the general solution using the method of undetermined coefficients

   i. $y'' + 2y' + 5y = 3\sin 2t$

   ii. $y'' + 4y' + 4y = 2e^{-2t}$

4. Solve the initial value problem $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$