MA 2051 D'20 Notes: Week 2

Integrating Factor Method

Integrating factors are used to solve the linear first-order differential equation of the form

$$y' + p(x)y = f(x). \tag{1}$$

NOTE: if the equation is not in the form (1) you will need to put it in such form before using this method.

- Step 1: Find the integrating factor: $\mu(x) = e^{\int p(x)\,dx}$, where $\int p(x)\,dx$ can be *any* antiderivative of $p(x)$ (typically we choose the arbitrary constant $C = 0$).
- Step 2: Multiply both sides by integrating factor to get $e^{\int p(x)\,dx}(y' + p(x)y) = f(x)e^{\int p(x)\,dx}$ which will *always* reduce to $\frac{d}{dx}\left(e^{\int p(x)\,dx}y(x)\right) = f(x)e^{\int p(x)\,dx}$.
- Step 3: Integrate the equation from Step 2 to get $e^{\int p(x)\,dx}y(x) = \int f(x)e^{\int p(x)\,dx}\,dx + C$.
- Step 4: Solve for $y(x)$ to get $y = e^{-\int p(x)\,dx}\int f(x)e^{\int p(x)\,dx}\,dx + Ce^{-\int p(x)\,dx}$.
- Step 5: If you are solving an initial value problem, use the initial condition to find the constant C.

Example: Find the general solution of $y' - 2xy = e^{x^2}$
This equation is in the form $y' + p(x)y = f(x)$ where $p(x) = -2x$ and $f(x) = e^{x^2}$.
Step 1: Find the integrating factor $\mu(x) = e^{\int p(x)\,dx} = e^{-\int 2x\,dx} = e^{-x^2}$
Step 2: Multiply both sides by integrating factor to get $\frac{d}{dx}\left(e^{-x^2}y(x)\right) = 1$
Step 3: Integrate the equation from Step 2 $e^{-x^2}y(x) = \int dx + c = x + c$
Step 4: Solve for $y(x)$, $y(x) = (x + c)e^{x^2}$.
Practice Problems: Find the solution to the given initial value problem.
1. $y' - y = 2te^{2t}$ and $y(0) = 1$ 2. $ty' + 2y = t^2 - t + 1$, where $y(1) = \dfrac{1}{2}$ and $t > 0$
3. $ty' + (t+1)y = t$, where $y(\ln 2) = 1$ and $t > 0$

Growth/Decay Phenomena

The **growth and decay equations** can be use to model many real-life growth and decay phenomena.

$$\frac{dy}{dt} = ky \text{ (growth)} \qquad \frac{dy}{dt} = -ky \text{ (decay)}, \quad \text{where } k > 0 \text{ is the constant of proportionality.}$$

Using the integrating factor method, we find the general solution $y(t) = ce^{kt}$ of the growth equation, known as the **exponential growth curve**. Similarly the **exponential decay curve** $y(t) = ce^{-kt}$ solves the decay equation. Alternatively, continuously compounded interest can be modeled by the equation $\frac{dS}{dt} = rS + d$, $S(0) = S_0$, where r is the interest rate and d is the amount of (yearly) deposit.
Using separation of variables, this has solution $S(t) = S_0e^{rt} + \frac{d}{r}\left(e^{rt} - 1\right)$.

Example: The number of bacteria in a colony increases at a rate proportional to the number present. If the number of bacteria triples every 10 hours, how long will it take for the colony to double in size.
The general solution of this problem is $y(t) = y_0e^{kt}$, where y_0 is the number of bacteria present initially.
By the question, we have $3y_0 = y_0e^{10k}$. So $k = \frac{\ln 3}{10}$, $y(t) = y_0e^{\frac{\ln 3}{10}t}$.
Then the time for the colony to double in size is the solution of $2y_0 = y_0e^{\frac{\ln 3}{10}t}$, $t = \frac{10\ln 2}{\ln 3} \approx 6.3$ hours.

Practice Problems:

1. A population model is used to calculate the number of cinnamon rolls on the plate in front of me. The rate of consumption $\frac{dC}{dt}$ is proportional to the amount on the plate. The model is $\frac{dC}{dt} = -kC$ where $k = 0.5$. Initially, at time $t = 0$, I had one dozen (12) cinnamon rolls.

a. Find the number of cinnamon rolls left at any time t.

b. Find the time $t_{1/2}$ where I have half of my original cinnamon rolls remaining.

Intro to Second-order Linear Equations

Any differential equation $y'' + p(x)y' + q(x)y = f(x)$, has an associated **homogeneous equation** obtained by simply replacing $f(x)$ with zero: $y'' + p(x)y' + q(x)y = 0$
Solving the differential equations can be broken down into two steps.

1. Find the *homogeneous solution*, denoted y_h

2. Find a *particular solution* (denoted y_p).

The **principle of superposition** states that any linear combination of homogeneous solutions will solve the homogeneous equation. Moreover, if y_h satisfies the homogeneous equation and y_p satisfies the nonhomogeneous equation, then $y_h + y_p$ solves the problem $y'' + p(x)y' + q(x)y = r(x)$.

Practice Problem: If $y_h(x) = c_1 e^t - c_2 e^{-t}$, $y_p(x) = -\sin(x)$ for the differential equation $y'' - y = 2\sin(x)$, show that $y_h(x)$ solves the homogeneous equation and that $y_p(x)$ and $y(x) = y_h(x) + y_p(x)$ solve the nonhomogeneous equation.

Characteristic Equation: Second Order

Suppose we have a linear, second order, constant coefficient differential equation $ay'' + by' + cy = 0$. As in the first order case, we assume $y = e^{rx}$. When we substitute this into the differential equation $ay'' + by' + cy = 0$ we get $ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$. As e^{rx} is never 0, it must be that

$$ar^2 + br + c = 0, \qquad \text{this is called } \textit{characteristic equation.}$$

Solving for r (which will have TWO solutions because this is a quadratic equation!) gives the *characteristic roots*. Depending on the type of root, we will have different solutions $y_1(x)$ and $y_2(x)$.
By the principle of superposition, $y_g(x) = C_1 y_1(x) + C_2 y_2(x)$.
Distinct Real Roots: If the two roots r_1, r_2 are real and distinct (two different numbers), then our solutions are

$$y_1(x) = e^{r_1 x}, \qquad y_2(x) = e^{r_2 x}$$

Real Double Root: If we have a double root r, like the solution of $(x-1)^2$, then our solutions are

$$y_1(x) = e^{rx}, \qquad y_2(x) = xe^{rx}$$

Complex Roots: If we have complex roots $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, like the solution of $x^2 + 1$, then our solutions (by Euler's formula) are $y_1(x) = e^{\alpha x} \sin \beta x, \qquad y_2(x) = e^{\alpha x} \cos \beta x$

Example (CASE 1): Find the general solution of $y'' + 2y' - 8y = 0$.
The characteristic equation is $r^2 + 2r - 8 = 0$ which has characteristic roots: $r_1 = 2,\ r_2 = -4$.
Then the general solution of the equation is $y_g(x) = C_1 e^{2x} + C_2 e^{-4x}$

Example (CASE 2): Find the general solution of $y'' + 6y' + 9y = 0$.
The characteristic equation is $r^2 + 6r + 9 = 0$ which has characteristic root: $r = 3$.
Then the general solution of the equation is $y_g(x) = C_1 e^{3x} + C_2 x e^{3x}$

Example (CASE 3): Find the general solution of $y'' + 2y'10y = 0$.
The characteristic equation is $r^2 + 2r + 10 = 0$ which has characteristic roots: $r_1 = -1 + 3i,\ r_2 = -1 - 3i$.
Then the general solution of the equation is $y_g(x) = C_1 e^{-x} \sin(3x) + C_2 e^{-x} \cos(3x)$

Practice Problems: Find the general solution to the following differential equations

1. $2y'' - 6y' - 8y = 0$ 2. $y'' = 169y$ 3. $9y'' + 6y' + y = 0$ 4. $y'' - 10y' + 26y = 0$