MA 2051 D'19 Notes: Week 1

Integration Review

Integration by Parts

Using the formula: $\int u\,dv = uv - \int v\,du$

Example: $\displaystyle\int x \sin x\,dx$

$u = x \qquad dv = \sin x\,dx \qquad du = dx \qquad v = -\cos x \qquad \displaystyle\int x\sin x\,dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + c$

Practice Problems: 1. $\displaystyle\int te^{-2t}\,dt$ 2. $\displaystyle\int x\cos 3x\,dx$

Partial Fraction Decomposition

Partial fraction decomposition is useful to integrate rational functions of polynomials in which the denominator has larger degree than the numerator. This will be very useful in more complicated problems that require separation of variables.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_k}{(ax+b)^k}, \quad k = 1, 2, \ldots$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}, \quad k = 1, 2, \ldots$

Example: $\displaystyle\int \frac{3x+9}{x^2+7x+10}\,dx$

$$\frac{3x+9}{x^2+7x+10} = \frac{A}{x+5} + \frac{B}{x+2} \rightarrow 3x+9 = A(x+2) + B(x+5)$$

Plug in $x = -5 \;\rightarrow -15+9 = A(-3) + B(0) \rightarrow A = 2$ Plug in $x = -2 \;\rightarrow -6+9 = A(0) + B(3) \rightarrow B = 1$

$$\int \frac{3x+9}{x^2+7x+10}dx = \int \frac{2}{x+5}dx + \int \frac{1}{x+2}dx = 2\ln|x+5| + \ln|x+2| + c$$

Practice Problems: 1. $\displaystyle\int \frac{1}{x^2-3x}\,dx$ 2. $\displaystyle\int \frac{x}{(x+1)(x^2+4)}\,dx$ 3. $\displaystyle\int \frac{2x}{x^2+4x+4}\,dx$

Classification of ODEs

Why is this important??? We need to classify ODEs in order to know which methods to use!

Order: The *order* of a differential equation is the highest derivative in the equation. For example, $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = e^t \sin t$ is a second order equation.

Linearity: An ODE is *linear* if it can be expressed in the form $A_1(t)y + A_2(t)\frac{dy}{dt} + \ldots + A_n(t)\frac{d^ny}{dt^n} = f(t)$. This means an equation is nonlinear if you have:

- Nonlinear functions of y or any of its derivatives (ex. $\sin\left(\frac{dy}{dx}\right)$ or y^2)
- Coefficients in front of y or any of its derivatives that depend on y.

Homogeneous vs. Inhomogeneous: If we have a linear equation in the form $A_1(t)y + A_2(t)\frac{dy}{dt} + \ldots + A_n(t)\frac{d^ny}{dt^n} = f(t)$, then the ODE is *homogeneous* if $f(t) \equiv 0$ and *inhomogeneous* otherwise.

Constant vs. Variable Coefficients: If we have a linear equation in the form $A_1(t)y + A_2(t)\frac{dy}{dt} + \ldots +$

$A_n(t)\frac{d^n y}{dt^n} = f(t)$, then the ODE has *constant* coefficients if $A_1(t), A_2(t), \ldots A_n(t)$ are all constant functions (i.e. numbers). Otherwise the ODE has *variable* coefficients.

Example: $\dfrac{d^2u}{d^2x} - \sin(x)u = 0$ is 2^{nd} order, linear, homogeneous, and has variable coefficients.

Practice Problems: 1. $\dfrac{d^2u}{dx^2} + x^3\dfrac{du}{dx} = \sin(x)u^2$ 2. $y''x^2 = xy$ 3. $\dfrac{dz}{dt} + 10\dfrac{d^3z}{dt^3} - z = t + \cos t$

Introduction to Differential Equations

Verifying Solutions to Differential Equations

Check Your Solutions! You should *never* get the wrong solution to a differential equation and not know it! You can easily check your answers by taking derivatives of your answer and subsistuting them in the ODE to see if your answer satisfies the equation.

Example Show that the function is a solution of the given differential equation:

$$\text{Differential equation: } \frac{1}{4}\left(\frac{d^2y}{dx^2}\right)^2 - x\frac{dy}{dx} + y = 1 - x^2, \qquad \text{Function: } y = x^2.$$

$$y = x^2, \qquad y' = 2x, \qquad y'' = 2$$

$$\frac{1}{4}\left(\frac{d^2y}{dx^2}\right)^2 - x\frac{dy}{dx} + y = \frac{1}{4}(2)^2 - x(2x) + x^2 = 1 - 2x^2 + x^2 = 1 - x^2$$

Practice Problems: 1. Differential Equation: $y'' - 9y = 0$, Function: $y(t) = e^{-3t}$
2. Differential Equation: $y'' - 2y' + y = (2 - 2t)\cos t - 2\sin t$, Function: $y(t) = t\sin t$

Verifying Implicit Solutions

Example: Show that the following relation defines an implicit solution to the given differential equation.

$$\text{Differential Equation: } 2xyy' = x^2 + y^2 \qquad \text{Relation: } y^2 = x^2 - cx$$

Beginning with the relation $y^2 = x^2 - cx$, differentiate implicitly giving: $2y\frac{dy}{dx} = 2x - c$ Notice that the left hand side is almost the same as that of the differential equation. Multiplying both sides by x gives: $2xyy' = 2x^2 - cx$ Observe that our equation has a c in it, whereas the differential equation does not. Returning to the original relation, however, we see that $cx = x^2 - y^2$. Substituting this gives: $2xyy' = 2x^2 - (x^2 - y^2) = x^2 + y^2$

Practice Problems: Show the following relation defines an implicit solution to the given differential equation. 1. Diff. Eq: $yy' = e^{2x}$ Rel: $y^2 = e^{2x}$ 2. Diff. Eq: $(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0$ Rel: $x + y + e^{xy} = 0$

Separable Equations

A first-order differential equation is *separable* if it can be written in the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$. We then solve this equation by multiplying both sides by $g(y)$, $g(y)\frac{dy}{dx} = f(x)$ and integrating both sides with respect to x:

$$\int g(y)\frac{dy}{dx}\,dx = \int g(y)\,dy = \int f(x)\,dx.$$

Example: Solve the initial value problem $\frac{dy}{dx} = \frac{\cos x}{y^2}; \quad y(0) = 2$

This equation is separable, with $f(x) = \cos x$ and $g(y) = y^2$. This gives: $\int y^2\,dy = \int \cos x\,dx$, therefore, $\frac{1}{3}y^3 = \sin x + c$. We use the inital value to solve for integration constant c. Substituting $x = 0$ and $y = 2$ gives $c = \frac{8}{3}$ This leads to the (implicit) solution: $y^3 = 3\sin x + 8$

Practice Problems: Determine which of the following equations is separable. If separable, find the general solution. 1. $y' = x^2/y$ 2. $y' + 2y = x/\cos(y)$ 3. $y' + y^2\sin x = 0$