COMPLEX NUMBERS

In algebra we encounter the problem of finding the roots of the polynomial

$$\lambda^2 + a\lambda + b = 0. \tag{1}$$

To find the roots, we use the quadratic formula to obtain

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}. \tag{2}$$

If $a^2 - 4b > 0$, there are two real roots. If $a^2 - 4b = 0$, we obtain the single root (of multiplicity 2) $\lambda = -\frac{a}{2}$. To deal with the case $a^2 - 4b < 0$, we introduce the **imaginary number***

Imaginary number

$$i = \sqrt{-1}. \tag{3}$$

Then for $a^2 - 4b < 0$,

$$\sqrt{a^2 - 4b} = \sqrt{(4b - a^2)(-1)} = \sqrt{4b - a^2}\,\sqrt{-1} = \sqrt{4b - a^2}\, i,$$

and the two roots of (1) are given by

$$\lambda_1 = -\frac{a}{2} + \frac{\sqrt{4b - a^2}}{2} i \quad \text{and} \quad \lambda_2 = -\frac{a}{2} - \frac{\sqrt{4b - a^2}}{2} i.$$

* You should not be troubled by the term "imaginary." It is just a name. The British mathematician Alfred North Whitehead, in the chapter on imaginary numbers in his *Introduction to Mathematics,* wrote:

> At this point it may be useful to observe that a certain type of intellect is always worrying itself and others by discussion as to the applicability of technical terms. Are the incommensurable numbers properly called numbers? Are the positive and negative numbers really numbers? Are the imaginary numbers imaginary, and are they numbers?—are types of such futile questions. Now, it cannot be too clearly understood that, in science, technical terms are names arbitrarily assigned, like Christian names to children. There can be no question of the names being right or wrong. They may be judicious or injudicious; for they can sometimes be so arranged as to be easy to remember, or so as to suggest relevant and important ideas. But the essential principle involved was quite clearly enunciated in Wonderland to Alice by Humpty Dumpty, when he told her, apropos of his use of words, 'I pay them extra and make them mean what I like'. So we will not bother as to whether imaginary numbers are imaginary, or as to whether they are numbers, but will take the phrase as the arbitrary name of a certain mathematical idea, which we will now endeavour to make plain.

EXAMPLE 1 ▶ **Finding roots**

Find the roots of the quadratic equation $\lambda^2 + 2\lambda + 5 = 0$.

Solution We have $a = 2$, $b = 5$, and $a^2 - 4b = -16$. Thus $\sqrt{a^2 - 4b} = \sqrt{-16} = \sqrt{16}\sqrt{-1} = 4i$, and the roots are

$$\lambda_1 = \frac{-2 + 4i}{2} = -1 + 2i \quad \text{and} \quad \lambda_2 = -1 - 2i. \quad ◀$$

Definition Complex number

A **complex number** is a number of the form

$$z = \alpha + i\beta, \tag{4}$$

where α and β are real numbers. α is called the **real part** of z and is denoted by Re z. β is called the **imaginary part** of z and is denoted by Im z. Representation **(4)** is sometimes called the **Cartesian form** of the complex number z.

Remark

If $\beta = 0$ in Equation **(4)**, then $z = \alpha$ is a real number. In this context we can regard the set of real numbers as a subset of the set of complex numbers.

EXAMPLE 2 ▶ **Real and imaginary parts**

In Example 1, Re $\lambda_1 = -1$ and Im $\lambda_1 = 2$. ◀

We can add and multiply complex numbers by using the standard rules of algebra.

EXAMPLE 3 ▶ **Adding, subtracting, and multiplying complex numbers**

Let $z = 2 + 3i$ and $w = 5 - 4i$. Calculate (a) $z + w$, (b) $3w - 5z$, and (c) zw.

Solution

(a) $z + w = (2 + 3i) + (5 - 4i) = (2 + 5) + (3 - 4)i = 7 - i$.

(b) $3w = 3(5 - 4i) = 15 - 12i$, $5z = 10 + 15i$, and
$3w - 5z = (15 - 12i) - (10 + 15i) = (15 - 10) + i(-12 - 15)$
$= 5 - 27i$.

(c) $zw = (2 + 3i)(5 - 4i) = (2)(5) + 2(-4i) + (3i)(5) + (3i)(-4i)$
$= 10 - 8i + 15i - 12i^2 = 10 + 7i + 12 = 22 + 7i$. Here we use the fact that $i^2 = -1$. ◀

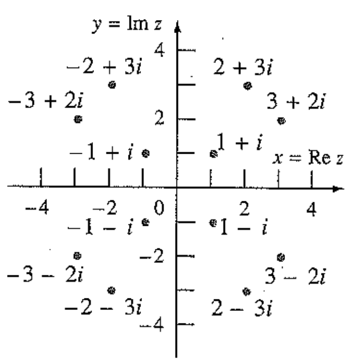

FIGURE A5.1

We can plot a complex number z in the xy-plane by plotting Re z along the x-axis and Im z along the y-axis. Thus each complex number can be thought of as a point in the xy-plane. With this representation the xy-plane is called the **complex plane.** Some representative points are plotted in Figure A5.1.

If $z = \alpha + i\beta$, then we define the **conjugate** of z, denoted $\bar{z}$, by

$$\bar{z} = \alpha - i\beta. \tag{5}$$

Figure A5.2 depicts a representative value of z and $\bar{z}$.

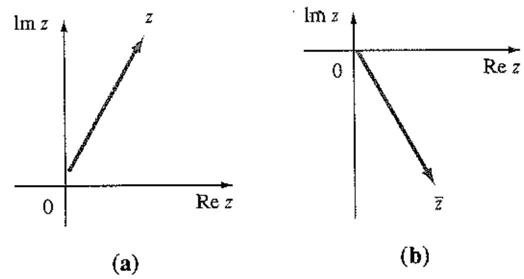

FIGURE A5.2

EXAMPLE 4 ▶ **Conjugate of a complex number**

Compute the conjugates of (a) $1 + i$, (b) $3 - 4i$, (c) $-7 + 5i$, and (d) -3.

Solution

(a) $\overline{1 + i} = 1 - i$. **(b)** $\overline{3 - 4i} = 3 + 4i$. **(c)** $\overline{-7 + 5i} = -7 - 5i$.
(d) $\overline{-3} = -3$. ◀

It is not difficult to show (see Problem 35) that

$$\bar{z} = z, \qquad \text{if and only if } z \text{ is real.} \tag{6}$$

If $z = \beta i$ with β real, then z is said to be **pure imaginary.** We can then show (see Problem 36) that

$$\bar{z} = -z, \qquad \text{if and only if } z \text{ is pure imaginary.} \tag{7}$$

Let $p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial with real coefficients. Then it can be shown (see Problem 41) that the complex roots of the equation $p_n(x) = 0$ occur in complex conjugate pairs; that is, if z is a root of $p_n(x) = 0$, then so is $\bar{z}$. We saw this fact illustrated in Example 1 in the case in which $n = 2$.

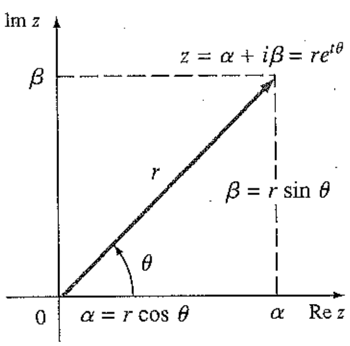

FIGURE A5.3

For $z = \alpha + i\beta$ we define the **magnitude** of z, denoted $|z|$, by

$$|z| = \sqrt{a^2 + \beta^2}, \tag{8}$$

and we define the **argument** of z, denoted by arg z, as the angle θ between the line $0z$ and the positive x-axis. From Figure A5.3 we see that $r = |z|$ is the distance from z to the origin, and, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$,

$$\theta = \arg z = \tan^{-1}\frac{\beta}{\alpha}. \tag{9}$$

By convention, we always choose values of arg z that lie in the interval

$$-\pi < \theta \le \pi. \tag{10}$$

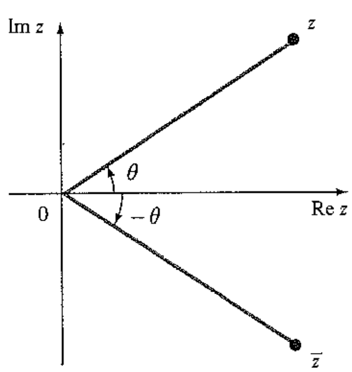

FIGURE A5.4

From Figure A5.4 we see that

$$|\bar{z}| = |z| \tag{11}$$

and

$$\arg \bar{z} = -\arg z. \tag{12}$$

We can use $|z|$ and arg z to describe what is often a more convenient way to represent complex numbers. From Figure A5.3 it is evident that, if $z = \alpha + i\beta$, $r = |z|$, and $\theta = \arg z$, then

$$\alpha = r\cos\theta \quad \text{and} \quad \beta = r\sin\theta. \tag{13}$$

We see at the end of this appendix that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{14}$$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, we also have

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta. \tag{14'}$$

Formula **(14)** is called the **Euler formula.** Using the Euler formula and Equations **(13),** we have

$$z = \alpha + i\beta = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta),$$

or

$$z = re^{i\theta}. \tag{15}$$

Representation **(15)** is called the **polar form** of the complex number z.

EXAMPLE 5 ▶ **Finding the polar form of complex numbers**

Determine the polar forms of the following complex numbers: (a) 1, (b) -1, (c) i, (d) $1 + i$, (e) $-1 - \sqrt{3}\,i$, and (f) $-2 + 7i$.

Solution The six points are plotted in Figure A5.5.

(a) From Figure A5.5(a) it is clear that $\arg 1 = 0$. Since $\text{Re}1 = 1$, we see that, in polar form,

$$1 = 1e^{i0} = 1e^{0} = e^{0}.$$

(b) Since $\arg(-1) = \pi$ [Figure A5.5(b)] and $|-1| = 1$, we have

$$-1 = 1e^{\pi i} = e^{i\pi}.$$

(c) From Figure A5.5(c) we see that $\arg i = \frac{\pi}{2}$. Since $|i| = \sqrt{0^2 + 1^2} = 1$, it follows that

$$i = e^{i\pi/2}.$$

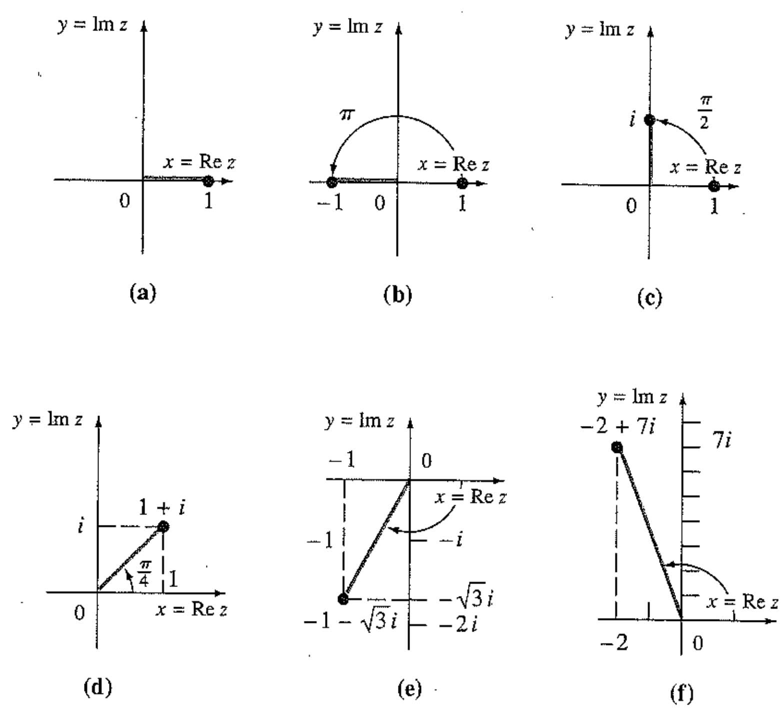

FIGURE A5.5

(d) $\arg(1+i) = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$, and $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$, so

$$1 + i = \sqrt{2}\, e^{i\pi/4}.$$

(e) Here $\tan^{-1}(\frac{\beta}{\alpha}) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$. However, $\arg z$ is in the third quadrant, so

$$\arg z = (\tfrac{\pi}{3}) - \pi = -\tfrac{2\pi}{3}.$$

Also, $|-1-\sqrt{3}\,i| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$, so

$$-1 - \sqrt{3}\, i = 2e^{-2\pi i/3}.$$

(f) To compute this complex number, we need a calculator. A calculator indicates that

$$\tan^{-1}(-\tfrac{7}{2}) = \tan^{-1}(-3.5) \approx -1.2925.$$

But $\tan^{-1} x$ is defined as a number in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Since from Figure A5.5(f) θ is in the second quadrant, we see that $\arg z = \tan^{-1}(-3.5) + \pi \approx 1.8491$. Next, we see that

$$|-2+7i| = \sqrt{(-2)^2+7^2} = \sqrt{53}.$$

Hence

$$-2 + 7i \approx \sqrt{53}\, e^{1.8491i}.$$ ◀

EXAMPLE 6 ▶ **Converting from polar to Cartesian form**

Convert the following complex numbers from polar to Cartesian form:
(a) $2e^{i\pi/3}$, **(b)** $4e^{3\pi i/2}$.

Solution

(a) $e^{i\pi/3} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)i$. Thus $2e^{i\pi/3} = 1 + \sqrt{3}\, i$.

(b) $e^{3\pi i/2} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = 0 + i(-1) = -i$. Thus $4e^{3\pi i/2} = -4i$.

◀

If $\theta = \arg z$, then by Equation **(12)**, $\arg \bar{z} = -\theta$. Thus, since $|\bar{z}| = |z|$, we have the following:

$$\text{If } z = re^{i\theta}, \quad \text{then } \bar{z} = re^{-i\theta}. \qquad \textbf{(16)}$$

Suppose we write a complex number in its polar form $z = re^{i\theta}$. Then

$$z^n = (re^{i\theta})^n = r^n(e^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta). \qquad \textbf{(17)}$$

Formula **(17)** is useful for a variety of computations. In particular, when $r = |z| = 1$, we obtain the

De Moivre formula*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \qquad \textbf{(18)}$$

*Abraham De Moivre (1667–1754) was a French mathematician well known for his work in probability theory, infinite series, and trigonometry. He was so highly regarded that Newton often told those who came to him with questions on mathematics, "Go to M. De Moivre; he knows these things better than I do."

EXAMPLE 7 ▶ **Using De Moivre's formula**

Compute $(1 + i)^5$.

Solution In Example 5(d) we showed that $1 + i = \sqrt{2}\, e^{\pi i/4}$. Then

$$(1 + i)^5 = (\sqrt{2}\, e^{\pi i/4})^5 = (\sqrt{2})^5 e^{5\pi i/4} = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4}\right)$$
$$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right) = -4 - 4i.$$

This can be checked by direct calculation. If the direct calculation seems no more difficult, then try to compute $(1 + i)^{20}$ directly. Proceeding as above, we obtain

$$(1 + i)^{20} = (\sqrt{2})^{20} e^{20\pi i/4} = 2^{10}(\cos 5\pi + i \sin 5\pi)$$
$$= 2^{10}(-1 + 0) = -1024.$$ ◀

Proof of Euler's formula

We now show that

$$e^{i\theta} = \cos\theta + i + \sin\theta \tag{19}$$

by using power series. We have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,^* \tag{20}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \tag{21}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots. \tag{22}$$

Then

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots. \tag{23}$$

Now $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and so on. Thus **(23)** can be written

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \ldots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$
$$= \cos\theta + i \sin\theta.$$

This completes the proof. ◆