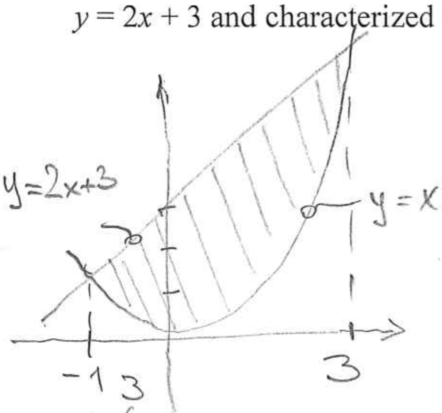
MA1024	TEST # 3	B'16

Section __B05Y__ Conference __BC5A | BC5B__ (Circle) NAME __Solutions__ (Print)

Point Total 100 (105 with *the bonus*)

1. (20 pts) Find the mass of the plane lamina of the shape of the region bounded by the parabola $y = x^2$ and the line $y = 2x + 3$ and characterized by the density $\delta(x, y)$ proportional to the distance between the point and the y-axis.



$x^2=2x+3$; $x^2-2x-3=0$; $(x+1)(x-3)=0 \to \begin{cases} x_1=-1 \\ x_2=3 \end{cases}$

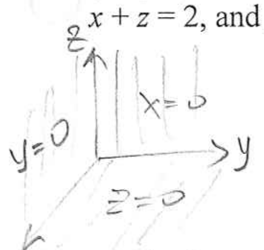
$\delta(x,y)=kx$

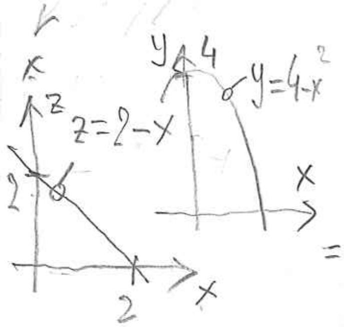
$$m=\int_{-1}^{3}\int_{x^2}^{2x+3} kx\,dy\,dx = k\int_{-1}^{3} xy\Big|_{x^2}^{2x+3}dx = k\int_{-1}^{3}\left[x(2x+3)-x\cdot x^2\right]dx =$$

$$=k\int_{-1}^{3}(3x+2x^2-x^3)\,dx = k\left[3\frac{x^2}{2}+2\frac{x^3}{3}-\frac{x^4}{4}\right]\Big|_{-1}^{3} = k\left[3\cdot\frac{3^2}{2}+2\frac{3^3}{3}-\frac{3^4}{4}-3\frac{(-1)^2}{2}-2\frac{(-1)^3}{3}+\frac{(-1)^4}{4}\right]=$$

$$=k\left[\frac{3\cdot9}{2}+\frac{2\cdot27}{3}-\frac{81}{4}-\frac{3}{2}+\frac{2}{3}+\frac{1}{4}\right]=k\,\frac{27\cdot6+54\cdot4-81\cdot3-18+8+3}{12}=k\,\frac{128}{12}=\frac{32}{3}k$$

2. (20 pts) Find the volume of the solid bounded by the graphs of the following functions: $x = 0$, $y = 0$, $z = 0$, $x + z = 2$, and $y = 4 - x^2 - z^2$.





$$V=\int_0^2\int_0^{2-x}\int_0^{4-x^2-z^2} dy\,dz\,dx = \int_0^2\int_0^{2-x} y\Big|_0^{4-x^2-z^2} dz\,dx = \int_0^2\int_0^{2-x}(4-x^2-z^2)\,dz\,dx =$$

$$=\int_0^2\left(4z-x^2z-\frac{z^3}{3}\right)\Big|_0^{2-x}dx = \int_0^2\left[4(2-x)-(2-x)x^2-\frac{(2-x)^3}{3}\right]dx =$$

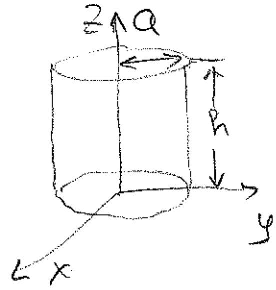
$(2-x)(2-x)^2=(2-x)(4-4x+x^2)=$
$=8-4x-8x+4x^2+2x^2-x^3=$
$=8-12x+6x^2-x^3$

$$=\int_0^2\left[8-4x-2x^2+x^3-\frac{8-12x+6x^2-x^3}{3}\right]dx=$$

$$=\int_0^2\frac{24-\cancel{12x}-6x^2+3x^3-8+\cancel{12x}-6x^2+x^3}{3}\,dx = \frac{1}{3}\int_0^2(16-12x^2+4x^3)\,dx =$$

$$=\frac{1}{3}\left(16x-\cancel{12}^{\,4}\frac{x^3}{\cancel{3}}+\cancel{4}\frac{x^4}{\cancel{4}}\right)\Big|_0^2=\frac{1}{3}(16\cdot2-4\cdot8+16)=\frac{48-32}{3}=\frac{16}{3}$$

3. (20 pts) Find the moment of inertia of the solid cylinder of radius a and height h around the axis of symmetry assuming that the density $\delta(x, y, z) = x^2 + y^2$.



Cylindrical coords: $0 \le r \le a$; $0 \le \theta \le 2\pi$; $0 \le z \le h$

$\delta = x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$;

$$I_z = \int_0^{2\pi}\int_0^{a}\int_0^{h} r^2 \cdot r^2 \cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{a} r^5 z\Big|_0^h dr\,d\theta = \int_0^{2\pi}\int_0^{a} r^5 h\,dr\,d\theta =$$

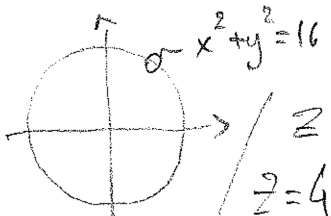
$$= \int_0^{a}\int_0^{2\pi} r^5 h\,d\theta\,dr = \int_0^{a} r^5 h 2\pi\,dr = 2\pi h\int_0^{a} r^5\,dr = 2\pi h\frac{r^6}{6}\Big|_0^a = \underline{\frac{1}{3}\pi h a^6}$$

4. (20 pts) Find the volume of the solid bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = 48 - x^2 - y^2$.

$z = 2x^2 + 2y^2 \Downarrow$ $\quad 2x^2 + 2y^2 = 48 - x^2 - y^2$; $\quad 3x^2 + 3y^2 = 48$; $\quad x^2 + y^2 = 16$

$z = 48 - x^2 - y^2 \Uparrow$ $\quad \Big|\ 0 \le r \le 4,\ 0 \le \theta \le 2\pi$

$$\Rightarrow V = \int_0^{2\pi}\int_0^{4}\int_{2r^2}^{48-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{4} rz\Big|_{2r^2}^{48-r^2} dr\,d\theta =$$



$z = 2(r\cos\theta)^2 + 2(r\sin\theta)^2 = 2r^2$

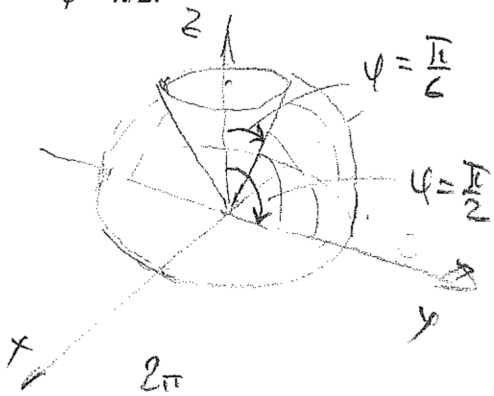
$z = 48 - (r\cos\theta)^2 - (r\sin\theta)^2 = 48 - r^2$

$$= \int_0^{2\pi}\int_0^{4}\left[r(48-r^2) - r(2r^2)\right]dr\,d\theta = \int_0^{2\pi}\int_0^{4}(48r - r^3 - 2r^3)\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^{4}(48r - 3r^3)\,dr\,d\theta = \int_0^{2\pi}\left[\frac{48r^2}{2} - \frac{3r^4}{4}\right]\Big|_0^4 d\theta = \int_0^{2\pi}\left[24\cdot 16 - \frac{3\cdot 16\cdot 16}{4}\right]d\theta =$$

$$= \int_0^{2\pi}(384 - 192) = \underline{384\pi}$$

5. (20 pts) Find the volume of the solid that is inside the sphere $\rho = 3$, below the cone $\phi = \pi/6$, and above the plane $\phi = \pi/2$.

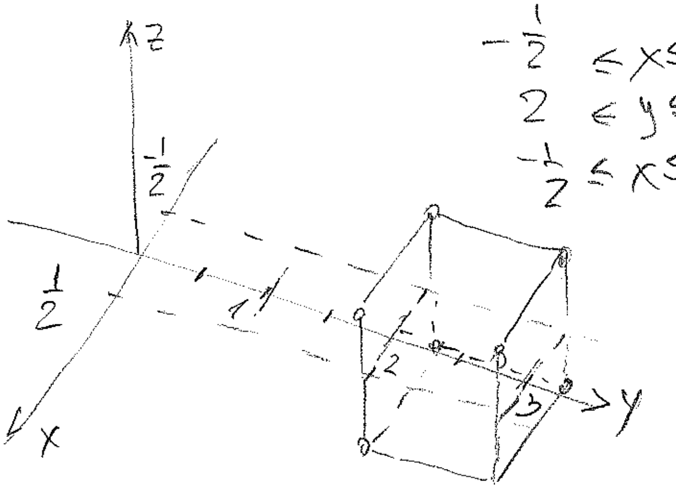


$$V = \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^3 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi}\int_{\pi/6}^{\pi/2} \frac{\rho^3}{3}\sin\varphi \Big|_0^3 d\varphi \, d\theta =$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/2} 9\sin\varphi \, d\varphi \, d\theta = 9\int_0^{2\pi} (-\cos\varphi)\Big|_{\pi/6}^{\pi/2} d\theta =$$

$$= 9\int_0^{2\pi}\Big(-\underbrace{\cos\frac{\pi}{2}}_{=0} + \underbrace{\cos\frac{\pi}{6}}_{=\frac{\sqrt{3}}{2}}\Big) d\theta = 9\frac{\sqrt{3}}{2}\int_0^{2\pi} d\theta = 9\frac{\sqrt{3}}{2}\,2\pi = \underline{9\pi\sqrt{3}}$$

6. (*5 bonus pts*) (**Do not address this until you have solved the regular problems 1-5**.) Derive a triple integral representing the moment of inertia about the *z*-axis of the cube with vertices $(\pm\frac{1}{2}, 2, \pm\frac{1}{2})$ and $(\pm\frac{1}{2}, 3, \pm\frac{1}{2})$.



$$-\frac{1}{2} \le x \le \frac{1}{2}$$
$$2 \le y \le 3$$
$$-\frac{1}{2} \le x \le \frac{1}{2}$$

$$I_z = \int_{-1/2}^{1/2}\int_2^3\int_{-1/2}^{1/2} (x^2+y^2)\, dx\, dy\, dz$$