Calculus III Primer

Powers and roots

EXPONENTS

Definition of Exponent

If n is a positive integer, then $x^n = \underbrace{x \cdot x \cdot x \cdots x}_{n \text{ factors}}$, and $x^0 = 1$; $x^{-n} = \frac{1}{x^n}$

If m and n are integers, then

$$x^{\frac{1}{n}} = \sqrt[n]{x} \text{ whenever } \sqrt[n]{x} \text{ is defined}$$

and

$$x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m \text{ whenever } \sqrt[n]{x} \text{ is defined}$$

Factorial numbers

$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ $\quad$ $0! = 1$ $\quad$ for n a nonnegative integer

ROOTS

Recall that for any positive *even* integer n (called the *index* of the radical) and any positive number x (called the *radicand*),

$$y = \sqrt[n]{x} \text{ if and only if } y > 0 \text{ and } y^n = x$$

We call y *the positive nth root of* x. For example, the positive fourth root of 16 is denoted by $\sqrt[4]{16}$; we write $\sqrt[4]{16} = 2$ since $2^4 = 16$.

For any positive *odd* integer n any number x (positive or negative),

$$y = \sqrt[n]{x} \text{ and only if } y^n = x$$

and y is called the *nth root* of x. For example, $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$.

Note that $\sqrt{x^2} = |x|$ for any number x.

LAWS OF EXPONENTS

If r and s are real numbers, then

$$x^r \cdot x^s = x^{r+s}$$

$$(x^r)^s = x^{rs} \text{ whenever } x^r \text{ is meaningful}$$

$$(xy)^r = x^r y^r \text{ whenever } x^r \text{ and } y^r \text{ are meaningful}$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r} \text{ whenever } x^r \text{ and } y^r \text{ are meaningful and } y^r \neq 0$$

$$\frac{x^r}{x^s} = x^{r-s} \text{ whenever } x^r \text{ and } x^s \text{ are meaningful and } x^s \neq 0$$

FACTORS AND EXPANSIONS

Difference of squares: $\quad a^2 - b^2 = (a - b)(a + b)$

Perfect square: $\quad (a + b)^2 = a^2 + 2ab + b^2$

LOGARITHMS

Properties of logarithms

$\log(MN) = \log M + \log N$

$\log(M/N) = \log M - \log N$

$\log M^n = n \log M$

$\log \sqrt[n]{M} = \frac{1}{n} \log M$

$\log_b b = 1$

$\log_n 1 = 0.$

Special bases:

$\log x = \log_{10} x$; this is called a common logarithm

$\ln x = \log_e x$ where $e \approx 2.71828182845905$; this is called a natural logarithm

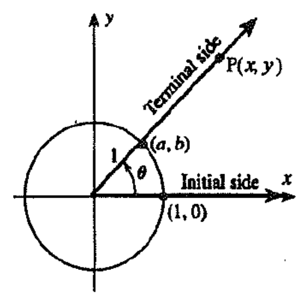
---

Trigonometric Functions

Let θ be any angle in standard position and let $P(x, y)$ be any point on the terminal side of the angle a distance of r from the origin ($r \neq 0$). Then

$\cos\theta = \frac{x}{r}$ $\quad \sin\theta = \frac{y}{r}$ $\quad \tan\theta = \frac{y}{x}$

$\sec\theta = \frac{r}{x}$ $\quad \csc\theta = \frac{r}{y}$ $\quad \cot\theta = \frac{x}{y}$



Standard position angle

RADIANS AND DEGREES

$360^\circ = 2\pi$ radians $= 1$ revolution

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \approx 57.2957\cdots \text{ degrees} \qquad 1 \text{ degree} = \left(\frac{\pi}{180}\right) \approx 0.0174532\cdots \text{ radian}$$

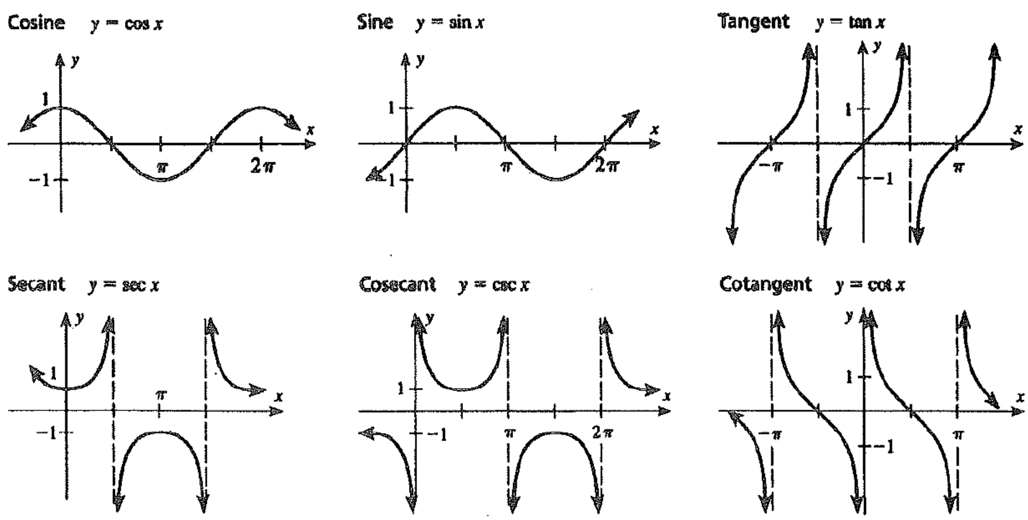
In calculus, it is often necessary to determine when a given function is not defined. For example, in Problem 17, Problem Set 1.3 in the textbook, for example, we need to know when $f(x) = 3\tan x - 5\sin x\cos x$ is not defined. From the definition given above, $f(\frac{\pi}{2})$ is not defined since $\tan x$ is not defined for $x = \frac{\pi}{2}$ (or for any multiple of $\frac{\pi}{2}$).

BY TABLE

Angle θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	q$\frac{3\pi}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.	0	undef.
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{2}}$	2	undef.	-1	undef.
$\csc\theta$	undef.	2	$\frac{2}{\sqrt{2}}$	$\frac{2}{\sqrt{3}}$	1	undef.	-1
$\cot\theta$	undef.	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	undef.	0

Trigonometric Graphs

TRIGONOMETRIC FUNCTIONS

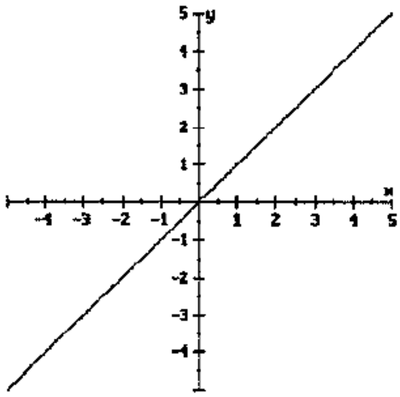


Inverse Trigonometric Functions

Inverse Function	Domain	Range
$y = \arccos x$ or $y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arcsin x$ or $y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arctan x$ or $y = \tan^{-1} x$	All reals	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{arccot}\, x$ or $y = \cot^{-1} x$	All reals	$0 < y < \pi$
$y = \text{arcsec}\, x$ or $y = \sec^{-1} x$	$\lvert x \rvert \geq 1$	$0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}$
$y = \text{arccsc}\, x$ or $y = \csc^{-1} x$	$\lvert x \rvert \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

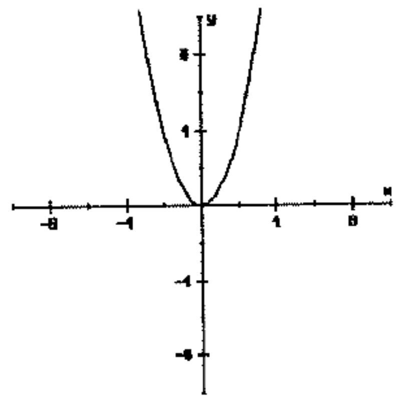
Graphs of Elementary Functions

Identity function



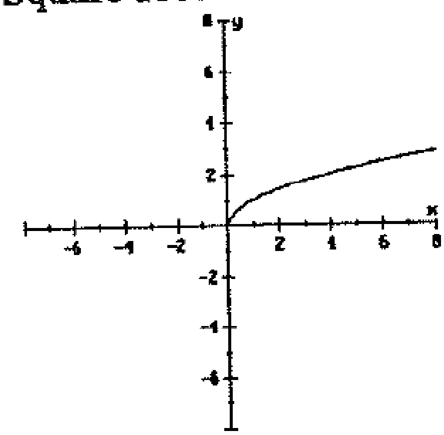
$y = x$

Parabola



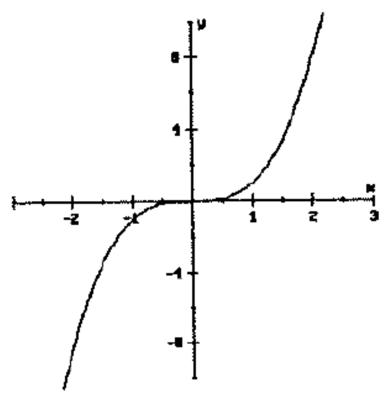
$y = ax^2 \qquad (a = 1)$

Square root function



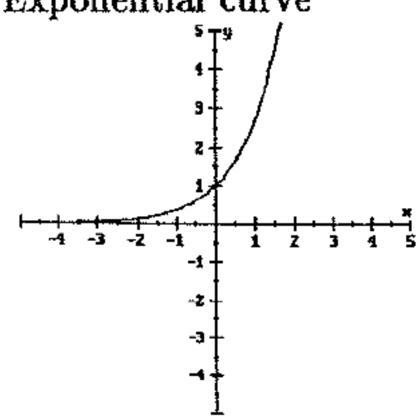
$y = a\sqrt{x} \qquad (a = 1)$

Cubic function



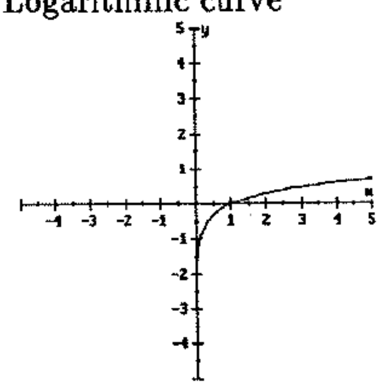
$y = ax^3 \qquad (a = 1)$

Exponential curve



$y = e^{ax} \qquad (a = 1)$

Logarithmic curve



$y = \log_a x \qquad (a = 10)$

Procedural Rules of Differentiation

If f and g are differentiable functions at x, u is a differentiable function of x, and a, b, and c are any real numbers, then the functions cf, $f + g$, fg, and f/g (for $g(x) \neq 0$) are also differentiable and their derivatives satisfy the following formulas:

Name of Rule	*Derivative*
1. **Constant multiple**	$(cf)' = cf'$
2. **Sum rule**	$(f + g)' = f' + g'$
3. **Difference rule**	$(f - g)' = f' - g'$
4. **Linearity rule**	$(af + bg)' = af' + bg'$
5. **Product rule**	$(fg)' = fg' + f'g$
6. **Quotient rule**	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$
7. **Chain rule**	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Differentiation Rules

EXTENDED POWER RULE

8. $\frac{d}{dx} u^n = nu^{n-1}\frac{du}{dx}$

TRIGONOMETRIC FUNCTIONS

9. $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

10. $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$

11. $\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$

12. $\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}$

13. $\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}$

14. $\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$

INVERSE TRIGONOMETRIC FUNCTIONS

15. $\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1 - u^2}}\frac{du}{dx}$

16. $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}$

17. $\frac{d}{dx}\tan^{-1} u = \frac{1}{1 + u^2}\frac{du}{dx}$

18. $\frac{d}{dx}\cot^{-1} u = \frac{-1}{1 + u^2}\frac{du}{dx}$

19. $\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}$

20. $\frac{d}{dx}\csc^{-1} u = \frac{-1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

21. $\frac{d}{dx}\ln|u| = \frac{1}{u}\frac{du}{dx}$

22. $\frac{d}{dx}\log_b|u| = \frac{\log_b e}{u}\frac{du}{dx} = \frac{1}{u \ln b}\frac{du}{dx}$

23. $\frac{d}{dx} e^u = e^u \frac{du}{dx}$

24. $\frac{d}{dx} b^u = b^u \ln b \frac{du}{dx}$

Elementary Integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1 \tag{1}$$

$$\int \frac{1}{u}\,du = \ln|u| + C \tag{2}$$

$$\int e^u\,du = e^u + C \tag{3}$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C \tag{4}$$

$$\int \cos u\,du = \sin u + C \tag{5}$$

$$\int \sin u\,du = -\cos u + C \tag{6}$$

$$\int \sec^2 u\,du = \tan u + C \tag{7}$$

$$\int \csc^2 u\,du = -\cot u + C \tag{8}$$

$$\int \sec u \tan u\,du = \sec u + C \tag{9}$$

$$\int \csc u \cot u\,du = -\csc u + C \tag{10}$$

$$\int \tan u\,du = \ln|\sec u| + C \tag{11}$$

$$\int \cot u\,du = \ln|\sin u| + C \tag{12}$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \tag{13}$$

$$\int \csc u\,du = -\ln|\csc u + \cot u| + C \tag{14}$$

$$\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \sin^{-1}\frac{u}{a} + C \tag{15}$$

$$\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \tag{16}$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \tag{17}$$