Supplements

Lecture Notes for the Reading Day

12/11

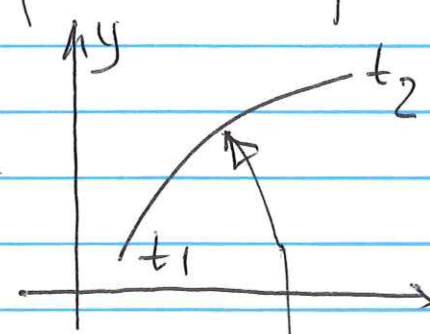
(13.1 & 13.2) 2. Motion of a Point in the Plane

Motion of a p. in the plane — an eloquent illustration of the use of vector-valued functions (VVF)

Reminder: • velocity — derivative of the position
• acceleration — derivative of the velocity

Coordinates of the moving p. — by parametric eqs.:

$x = f_1(t)$
$y = f_2(t)$, where t is time



The position is described by a vector function (called the position vector or displacement)

graph of a position vector: trajectory

$\vec{r}(t) = f_1(t)\vec{i} + f_2(t)\vec{j}$, and:

velocity: $\vec{v}(t) = \vec{r}\,'(t) = f_1'(t)\vec{i} + f_2'(t)\vec{j}$

acceleration: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = f_1''(t)\vec{i} + f_2''(t)\vec{j}$

The magnitude of the velocity vector:

$|\vec{v}|$ – speed (scalar!)

Example 1: The vector $\vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j}$ gives the position of a moving object; we look for: (a) acceleration; (b) the speed when $t=2$

Solution: $\vec{v} = \vec{r}'(t) = -(3\sin t)\vec{i} + (3\cos t)\vec{j}$

$\vec{a} = \vec{r}''(t) = -(3\cos t)\vec{i} - (3\sin t)\vec{j} \rightarrow$ (a)!

At $t=2$; the speed is:

$$|\vec{v}(2)| = \sqrt{(-3\sin 2)^2 + (3\cos 2)^2} = 3 \rightarrow \text{(b)!}$$

Example 2: The vector $\vec{v} = \vec{r}'(t) = (\cos t)\vec{i} - (\sin t)\vec{j}$ gives the velocity of a moving object; we look for the position if $\vec{r} = 2\vec{i}$ when $t=0$.

Solution: Integrating both side w.r.to t:

$$r(t) = (\sin t)\vec{i} + (\cos t)\vec{j} + \vec{C}$$

To find $\vec{c}$, we use the initial condition. The initial moment of motion is $t=0$, so:

$(\sin 0)\vec{i} + (\cos 0)\vec{j} + \vec{c} = 2\vec{i}$; $\vec{C} = 2\vec{i} - \vec{j}$, and the solution is:

$$\vec{r}(t) = (\sin t + 2)\vec{i} + (\cos t - 1)\vec{j}$$

The length of a smooth curve

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k},$$

where $\alpha \le t \le \beta$,

is traced as t goes from $t=\alpha$ to $t=\beta$, and

$$L = \int_{\alpha}^{\beta} |\vec{v}|\, dt$$

Example: Find the length of the curve

$$\vec{r}(t) = 6t^3\,\vec{i} + 2t^3\,\vec{j} - 3t^3\,\vec{k} \text{ if } 1 \le t \le 2.$$

Solution: From $\vec{r}(t)$ to $\vec{v}(t)$:

$$\vec{v} = \vec{r}\,'(t) = 18t^2\,\vec{i} - 6t^2\,\vec{j} - 9t^2\,\vec{k}, \text{ and}$$

$$|\vec{v}| = \sqrt{(18t^2)^2 + (-6t^2)^2 + (-9t^2)^2} = \sqrt{441t^4} = 21t^2$$

$$L = \int_1^2 21t^2\, dt = 7t^3 \Big|_1^2 = 49 \quad //$$