Supplements

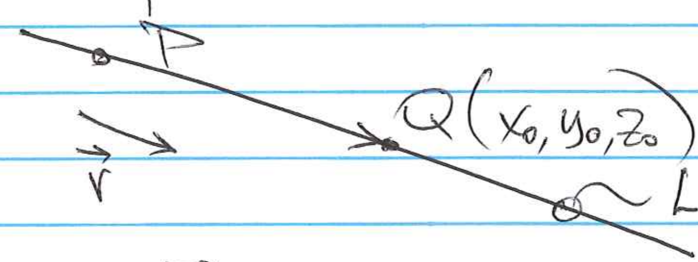
12/11

Lecture Notes for the Reading Day

(12.5) 1. Lines & Planes in Space

Concepts & methods of Vector Calc — for characterizing Lines & Planes

Line L in space may be completely determined by one of its points and its direction!



- L - is a line in space
- L contains $Q(x_0, y_0, z_0)$
- L is $\parallel$ to vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$
- P is any p. on L $\Rightarrow$ $\overrightarrow{PQ}$ is parallel to $\vec{v}$:

$$\overrightarrow{PQ} = t\vec{v}, \text{ or: } (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k} =$$
$$= t[a\vec{i} + b\vec{j} + c\vec{k}] \quad (*)$$

The components of both sides of the eq. (*) should be equal each other, so:

Def: Parametric Form of a Line in R^3 $\Longrightarrow$

<u>Def.</u>: The p. $P(x, y, z)$ is on the line L if its coordinates satisfy

$$x = x_0 + ta; \quad y = y_0 + tb; \quad z = z_0 + tc;$$

where t is some number; a, b, c are the components of the aligned vector $\vec{v} = \langle a, b, c \rangle$, that indicates the direction of the line (they are also called <u>Direction numbers</u>); x_0, y_0, z_0 are the coordinates of some known p. Q on the line

<u>Example</u>: [Knowing a p. on L, ~~k~~knowing direction of L, find Parametric Form of L]

Given the line containing p. $Q(3, 1, 4)$ and aligning with the vector $\vec{v} = -\vec{i} + \vec{j} - 2\vec{k}$ (that is, vector $\vec{v}$ is $\parallel$ to the line), formulate the equations of the parametric form of the line.

<u>Solution</u>: $a = -1$; $b = 1$; $c = -2$; $x_0 = 3$; $y_0 = 1$; $z_0 = 4$;

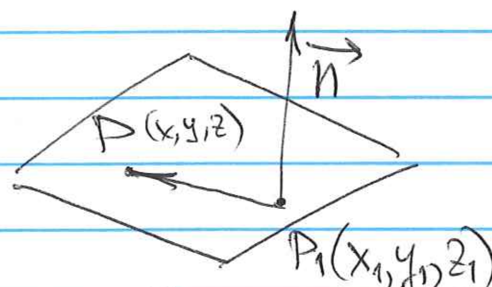
so: $x = 3 - t$; $y = 1 + t$; $z = 4 - 2t$. $*$

Plane Conside vector $\vec{n} = \langle A, B, C \rangle$ (+)
with initial p. $P_1(x_1, y_1, z_1)$

Recall: if we perform the dot product of two orthogonal vectors, the result is 0.

Assume that P_1 is located in the plane, and consider also an arbitrary p. on the same plane P with the coordinates (x, y, z). Also, assume $\vec{n}$ to be normal (i.e. perpendicular, or orthogonal) to the plane. Then

$$\overrightarrow{P_1P} \cdot \vec{n} = 0$$



We thus can say: the set of all points P satisfying this eq. is the plane through P_1 and perpendicular (normal) to $\vec{n}$

Vector n is called the normal vector to the plane.

If $\overrightarrow{P_1P} = \langle x - x_1, y - y_1, z - z_1 \rangle$ (++),
then the Cartesian eq. of such a plane is:

$$\boxed{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0} \quad (+++)$$

This is the standard form for the eq. of a plane.

From (++) (by regrouping the terms):

$$Ax + By + Cz = D \quad (A)$$

called the <u>general form</u> of the eq. of a plane in space.

Eq. (A) is a linear eq.! – every plane has a linear eq.; and inverse is also true: the graph of a linear eq. is a plane

<u>Example 1</u>: Find the eq. of the plane passing through the p. $(3, -2, 1)$ normal to the vector $\vec{n} = 2\vec{i} + \vec{j} + \vec{k}$.

Solution: $(2)(x-3) + (1)(y-(-2)) + (1)(z-1) = 0 \Rightarrow$

$$\Rightarrow 2x - 6 + y + 2 + z - 1 = 0 \Rightarrow \underline{2x + y + z = 5} \quad (*)$$

<u>Example 2</u>: Find the angle between the planes:

$x - 2y + z = 0$ & $2x + 3y - 2z = 0$

Solution: $\vec{n_1} = \langle 1, -2, 1 \rangle$ & $\vec{n_2} = \langle 2, 3, -2 \rangle$;

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{2-6-2}{\sqrt{1^2+(-2)^2+1^2}\sqrt{2^2+3^2+(-2)^2}} =$$

$$= \frac{-6}{\sqrt{6}\sqrt{17}} = \frac{-6}{\sqrt{102}} \simeq -0.594; \quad \theta = \cos^{-1}(-0.594) \cong \underline{126.4^\circ}$$