MA1022:
Calculus II

Supplementary Class Notes

PART III

B’18
2019-2020

Elementary Integrals: Extended List – Part 1

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1 \tag{1}$$

$$\int \frac{1}{u}\,du = \ln|u| + C \tag{2}$$

$$\int e^u\,du = e^u + C \tag{3}$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C \tag{4}$$

$$\int \cos u\,du = \sin u + C \tag{5}$$

$$\int \sin u\,du = -\cos u + C \tag{6}$$

$$\int \sec^2 u\,du = \tan u + C \tag{7}$$

$$\int \csc^2 u\,du = -\cot u + C \tag{8}$$

$$\int \sec u \tan u\,du = \sec u + C \tag{9}$$

$$\int \csc u \cot u\,du = -\csc u + C \tag{10}$$

Elementary Integrals: Extended List – Part 2

$$\int \tan u\,du = \ln\left|\sec u\right| + C \qquad (11)$$

$$\int \cot u\,du = \ln\left|\sin u\right| + C \qquad (12)$$

$$\int \sec u\,du = \ln\left|\sec u + \tan u\right| + C \qquad (13)$$

$$\int \csc u\,du = -\ln\left|\csc u + \cot u\right| + C \qquad (14)$$

$$\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \sin^{-1}\frac{u}{a} + C \qquad (15)$$

$$\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \qquad (16)$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad (17)$$

Algebraic Procedures for Transforming Integrands

Technique	*Example*	#
Expand (numerator)	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$	
Separate numerator	$\dfrac{1+x}{x^2+1} = \dfrac{1}{x^2+1} + \dfrac{x}{x^2+1}$	5
Complete the square	$\dfrac{1}{\sqrt{2x - x^2}} = \dfrac{1}{\sqrt{1 - (x-1)^2}}$	3
Divide improper rational function	$\dfrac{x^2}{x^2+1} = 1 - \dfrac{1}{x^2+1}$	8
Add and subtract terms in numerator	$\dfrac{2x}{x^2+2x+1} = \dfrac{2x+2-2}{x^2+2x+1} = \dfrac{2x+2}{x^2+2x+1} - \dfrac{2}{(x+1)^2}$	8

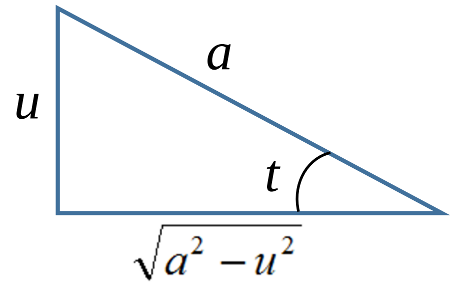
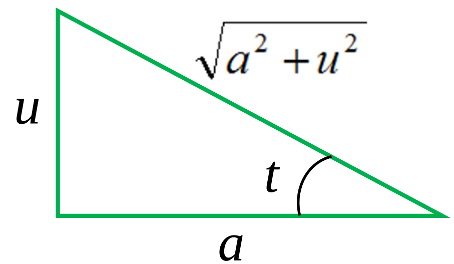
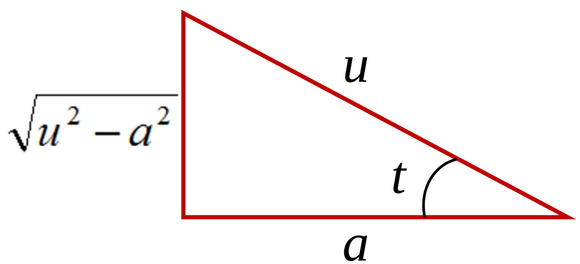
DO NOT SEPARATE DENOMINATORS!

$$\frac{1}{x^2+1} \neq \frac{1}{x^2} + \frac{1}{1}$$

Trigonometric Identities for Trigonometric Integrals

Integral	***Technique***
1. $\int \sin^n x\,dx$ & $\int \cos^n x\,dx$	
1a. n is odd	$\sin^2 x + \cos^2 x = 1$
1b. n is even	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
2. $\int \sin^m x \cos^n x\,dx$	
2a. n or m is odd	$\sin^2 x + \cos^2 x = 1$
2b. both m and n are even	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
3. $\int \tan^n x\,dx$ & $\int \cot^n x\,dx$	$\tan^2 x = \sec^2 x - 1$, $\cot^2 x = \csc^2 x - 1$
4. $\int \tan^m x \sec^n x\,dx$ & $\int \cot^m x \csc^n x\,dx$	
4a. n even, m any number	$\sec^2 x = \tan^2 x + 1$
4b. m odd, n any number	$\tan^2 x = \sec^2 x - 1$, $\cot^2 x = \csc^2 x - 1$

Trigonometric Substitutions

Integrand's Fraction		Substitution (Restriction)
(1) $\sqrt{a^2 - u^2}$		$u = a\sin t$ $(-\pi/2 \le t \le \pi/2)$
(2) $\sqrt{a^2 + u^2}$		$u = a\tan t$ $(-\pi/2 \le t \le \pi/2)$
(3) $\sqrt{u^2 - a^2}$		$u = a\sec t$ $(0 \le t \le \pi, t \ne \pi/2)$

Partial Fractions (1)

… with Simple Linear Factors – Examples (0) & (A)

Suppose $f(x) = p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q. Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1. Factor the denominator q in the form $(x - r_1)(x - r_2) \cdots (x - r_n)$, where $r_1, \ldots, r_n$ are real numbers.

Step 2. Partial fraction decomposition Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

Step 3. Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$, which produces conditions for $A_1, \ldots, A_n$.

Step 4. Solve for coefficients Equate like powers of x in Step 3 to solve for the undetermined coefficients $A_1, \ldots, A_n$.

Partial Fractions (2)

… for Repeated Linear Factors

Suppose the repeated linear factor $(x - r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x - r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_m}{(x - r)^m},$$

where $A_1, \ldots, A_m$ are constants to be determined.

… with Simple Irreducible Quadratic Factors – Example (B)

Suppose a simple irreducible factor $ax^2 + bx + c$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

where A and B are unknown coefficients to be determined.