MA1021 INTERMEDIATE EXAM # 1 A'16

Section A06 NAME ______________________

Print

Point Total 100 (105 with the *bonus*)

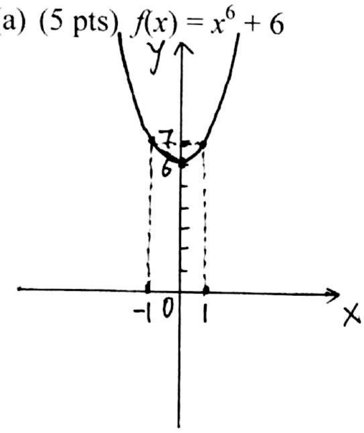
1. (4 pts) Find the domain of the function $f(x)=\sqrt{\frac{2}{3-x}}$.

$3-x \neq 0$ and $\frac{2}{3-x} \geq 0 \Rightarrow x \neq 3$ and $x \leq 3 \Rightarrow x<3$.

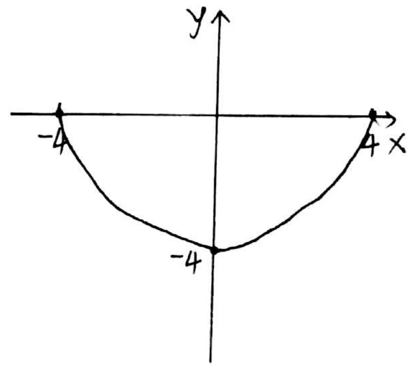
So the domain of $f(x)$ is $(-\infty, 3)$.

2. Sketch the graphs of the following functions:

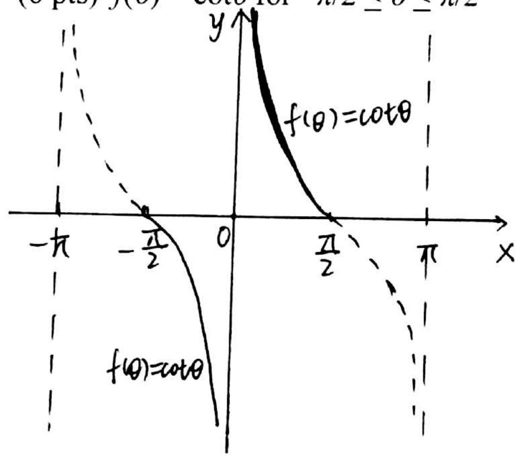
(a) (5 pts) $f(x) = x^6 + 6$



(b) (5 pts) $f(x) = -\sqrt{16-x^2}$



(6 pts) $f(\theta) = \cot\theta$ for $-\pi/2 \leq \theta \leq \pi/2$

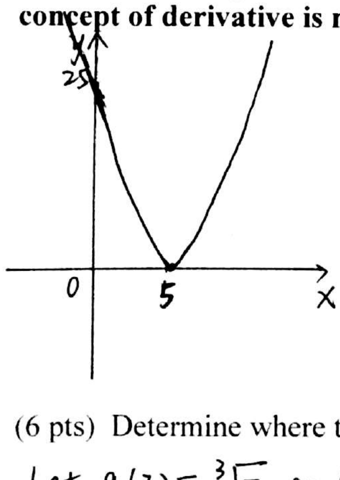


3. (5 pts) Find $f(g(x))$ and $g(f(x))$ if $f(x) = x^2 - 8$ and $g(x)=\sqrt{x+8}$.

(1) $f(g(x)) = [g(x)]^2 - 8 = (\sqrt{x+8})^2 - 8 = x+8-8 = x$

(2) $g(f(x)) = \sqrt{f(x)+8} = \sqrt{x^2-8+8} = \sqrt{x^2} = |x|$.

4. (16 pts) Find all points of the curve $y = (x-5)^2$ at which the tangent line is horizontal. (**Note: The use of the concept of derivative is not eligible.**)



If the tangent line is horizontal, then the slope of this line is 0, i.e. $m=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{h\to 0}\frac{(x+h-5)^2-(x-5)^2}{h}=0$.

We have $\frac{(x+h-5)^2-(x-5)^2}{h} = \frac{(x+h)^2-10(x+h)+25-(x^2-10x+25)}{h}$

$= \frac{x^2+2hx+h^2-10x-10h+25-x^2+10x-25}{h} = \frac{2hx+h^2-10h}{h} = 2x+h-10$

Hence, $0 = \lim\limits_{h\to 0}\frac{(x+h-5)^2-(x-5)^2}{h}$

$= \lim\limits_{h\to 0}(2x+h-10) = 2x-10$

$\Rightarrow x=5$. When $x=5$, $y=(x-5)^2=0$.

Therefore, $(5,0)$ is the only point at which the tangent line is horizontal.

5. (6 pts) Determine where the function $f(z)=\sqrt[3]{3-z^3}$ is continuous.

Let $g(z)=\sqrt[3]{z}$ and $h(z)=3-z^3$, then $f(z)=g(h(z))$.

$h(z)$, as a polynomial, is continuous on $(-\infty,\infty)$;

and $g(z)$, as a power function, is continuous on $(-\infty,\infty)$.

Thus, $f(z)$, as a composite function of $h(z)$ and $g(z)$, is continuous on $(-\infty,\infty)$.

6. Evaluate the limits:

(a) (15 pts) $\lim\limits_{y\to 0}\dfrac{\sqrt{3y+4}-2}{5y}$ $= \lim\limits_{y\to 0}\dfrac{\sqrt{3y+4}-2}{5y}\cdot\dfrac{\sqrt{3y+4}+2}{\sqrt{3y+4}+2} = \lim\limits_{y\to 0}\dfrac{(\sqrt{3y+4})^2-2^2}{5y(\sqrt{3y+4}+2)}$

$= \lim\limits_{y\to 0}\dfrac{3y+4-4}{5y(\sqrt{3y+4}+2)} = \lim\limits_{y\to 0}\dfrac{3y}{5y(\sqrt{3y+4}+2)} = \lim\limits_{y\to 0}\dfrac{3}{5(\sqrt{3y+4}+2)} = \dfrac{3}{5(\sqrt{0+4}+2)}$

$= \dfrac{3}{5(2+2)} = \dfrac{3}{20}$

(b) (8 pts) $\lim\limits_{x\to 0}\dfrac{3x^3+9x^2}{5x^4-18x^2}$

$= \lim\limits_{x\to 0}\dfrac{x^2(3x+9)}{x^2(5x^2-18)} = \lim\limits_{x\to 0}\dfrac{3x+9}{5x^2-18} = \dfrac{\lim\limits_{x\to 0}(3x+9)}{\lim\limits_{x\to 0}(5x^2-18)} = \dfrac{0+9}{0-18} = -\dfrac{1}{2}.$

(c) (10 pts) $\lim\limits_{x\to 1}\dfrac{1+x+4\cos x+\frac{1}{2}\tan\frac{\pi x}{4}}{4\sin x - x\cos x}$ $= \dfrac{1+1+4\cos 1+\frac{1}{2}\tan\frac{\pi}{4}}{4\sin 1-\cos 1} = \dfrac{2+4\cos 1+\frac{1}{2}\cdot 1}{4\sin 1-\cos 1} = \dfrac{\frac{5}{2}+4\cos 1}{4\sin 1-\cos 1}$

(Note: $\cos 1 \neq 0$, $\sin 1 \neq 0$, $\tan\frac{\pi}{4}=1$.)

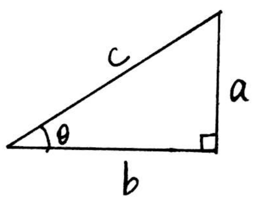
(d) (8 pts) $\lim\limits_{t\to 0}\dfrac{\sin t}{\sin 4t}$ $= \lim\limits_{t\to 0}\dfrac{\sin t}{\sin 4t}\cdot\dfrac{4t}{4t} = \lim\limits_{t\to 0}\dfrac{\sin t}{t}\cdot\dfrac{4t}{\sin 4t}\cdot\dfrac{1}{4} = \dfrac{1}{4}\left(\lim\limits_{t\to 0}\dfrac{\sin t}{t}\right)\cdot\left(\lim\limits_{t\to 0}\dfrac{4t}{\sin 4t}\right)$

$= \dfrac{1}{4}\times 1\times 1 = \dfrac{1}{4}$, since $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = \lim\limits_{\theta\to 0}\dfrac{\theta}{\sin\theta} = 1$. (Note: $\sin 4t \neq 4\sin t$.)

(e) (12 pts) $\lim\limits_{x\to\infty}\dfrac{40x^4+4x^2-1}{10x^4+8x^2+1}$

$= \lim\limits_{x\to\infty}\dfrac{\frac{40x^4}{x^4}+\frac{4x^2}{x^4}-\frac{1}{x^4}}{\frac{10x^4}{x^4}+\frac{8x^2}{x^4}+\frac{1}{x^4}} = \lim\limits_{x\to\infty}\dfrac{40+\frac{4}{x^2}-\frac{1}{x^4}}{10+\frac{8}{x^2}+\frac{1}{x^4}} = \dfrac{40+0-0}{10+0+0} = 4$, since $\lim\limits_{x\to\infty}\dfrac{1}{x^2}=0$ and $\lim\limits_{x\to\infty}\dfrac{1}{x^4}=0$.

7. (*5 bonus pts*) (**<u>Note</u>: Do not address this until you have solved all the problems above!**) Simplify the expression $\sin(\tan^{-1}x)$.



(Figure 1)

Let $\theta = \tan^{-1}x$, then $\tan(\theta) = \tan(\tan^{-1}x) = x$.

Refer to Figure 1, we have $\tan\theta = \dfrac{a}{b}$, so $\dfrac{a}{b} = x$.

Take $b=1$, then $a = bx = x \Rightarrow c = \sqrt{a^2+b^2} = \sqrt{x^2+1}$.

Hence, $\sin(\tan^{-1}x) = \sin\theta = \dfrac{a}{c} = \dfrac{x}{\sqrt{x^2+1}}$.