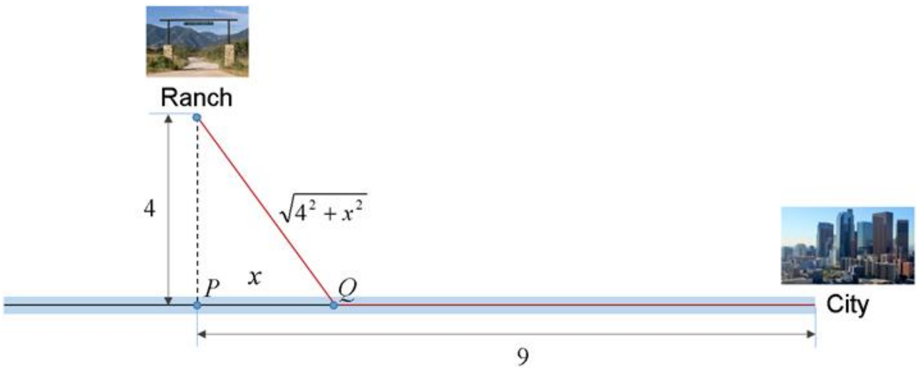
Additional Class Notes

October 11, 2016

Applied Optimization

Problem 2.

Cowboy Clint finished building the enclosure for his horses, and now he wants to build a dirt road from his ranch to the highway so that he can drive to the city in the shortest amount of time. What he knows: distance from the ranch to the highway is 4 mi, and distance to the city down the highway is 9 mi.



Where should Clint join the dirt road to the highway if the speed limit is 20 mph on the dirt road and 55 mph on the highway?

Solution

Step 1: *Choose variables*. Actually, the question here is where the dirt load should join the highway, in other words, where to put the point Q. Let P be the point on the highway nearest the ranch and let x be the distance from P to the point Q where the dirt road joins the highway.

Step 2: *Find the function and the interval*. We want to minimize the ***travel time***, so we need to compute the travel time $T(x)$ of the trip as a function of x. The length of the dirt road is therefore $\sqrt{4^2+x^2}$.

As *time* = *distance*/*speed*, then:

- to travel the dirt road at 20 mph, it will take: $t_1 = \sqrt{4^2+x^2}/20$
- to travel the highway at 55 mph, it will take: $t_2 = (9-x)/55$

Total time therefore is:

$$T(x) = t_1 + t_2 = \sqrt{4^2+x^2}/20 + (9-x)/55$$

Interval for optimization is the interval of possible values of x; it is obviously

$0 \le x \le 9$.

Step 3: *Optimize the function*. We solve $T'(x) = 0$ to find critical point(s):

$$T'(x) = \frac{x}{20\sqrt{16+x^2}} - \frac{1}{55}$$

$$55x = 20\sqrt{16+x^2}$$

$$121x^2 = 16\sqrt{16+x^2}$$

$$x = 16/\sqrt{105} \approx 1.56 \text{ [mi]}$$

This value of x is the only critical point. To find the minimum value of the travel time, we compute $T(x)$ at the critical point and end points of [0,9]:

$T(0) \approx 0.36$ [h]; $T(1.56) \approx 0.35$ [h]; $T(9) \approx 0.49$ [h]

The travel time is therefore minimized if the dirt road joins the highway at $16/\sqrt{105} \approx 1.56$ miles.