Exercise 1.6

32. $f(x)=\dfrac{\sqrt{x}}{\sqrt{x}-3}$. Find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

(a) To find $f^{-1}(x)$:

Step 1: Solve for x in terms of y.

$$y=\frac{\sqrt{x}}{\sqrt{x}-3} \Rightarrow y(\sqrt{x}-3)=\sqrt{x} \Rightarrow y\sqrt{x}-3y=\sqrt{x} \Rightarrow (y-1)\sqrt{x}=3y$$

$$\Rightarrow \sqrt{x}=\frac{3y}{y-1} \Rightarrow x=\left(\frac{3y}{y-1}\right)^2$$

Step 2: Interchange x and y.

$$y=\left(\frac{3x}{x-1}\right)^2=f^{-1}(x)$$

(b) To find the domain and range of $f^{-1}(x)$:

The domain of $f(x)$ is: $x\geq 0$ and $x\neq 9$ (since $\sqrt{x}-3\neq 0$), i.e. $(9,\infty)\cup[0,9)$, so the range of $f^{-1}(x)$ is also $(9,\infty)\cup[0,9)$.

If $0\leq x<9$, then $0\leq\sqrt{x}<3 \Rightarrow \sqrt{x}-3<0$. Hence $f(x)=\dfrac{\sqrt{x}}{\sqrt{x}-3}\leq 0$.

If $x>9$, then $\sqrt{x}>0$ and $\sqrt{x}-3>0 \Rightarrow \sqrt{x}>\sqrt{x}-3>0$. Hence $f(x)=\dfrac{\sqrt{x}}{\sqrt{x}-3}>1$.

Therefore, the range of $f(x)$ is $(-\infty,0]\cup(1,\infty)$, which means that the domain of $f^{-1}(x)$ is also $(-\infty,0]\cup(1,\infty)$.

(c) To show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$:

$$f(f^{-1}(x))=\frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3}.$$

On the domain of $f^{-1}(x)$ $(-\infty,0]\cup(1,\infty)$, we have $\dfrac{3x}{x-1}\geq 0$, so

$$f(f^{-1}(x))=\frac{\frac{3x}{x-1}}{\frac{3x}{x-1}-3}=\frac{3x}{3x-3(x-1)}=\frac{3x}{3}=x.$$

And $f^{-1}(f(x))=\left(\dfrac{3\frac{\sqrt{x}}{\sqrt{x}-3}}{\frac{\sqrt{x}}{\sqrt{x}-3}-1}\right)^2=\left(\dfrac{3\sqrt{x}}{\sqrt{x}-(\sqrt{x}-3)}\right)^2=x.$