On the complexity of computing discrete logarithms in the field $\mathbb{F}_{3^{6\cdot 509}}$

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Joint work with:

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Alfred Menezes	University of Waterloo
Thomaz Oliveira	CINVESTAV-IPN

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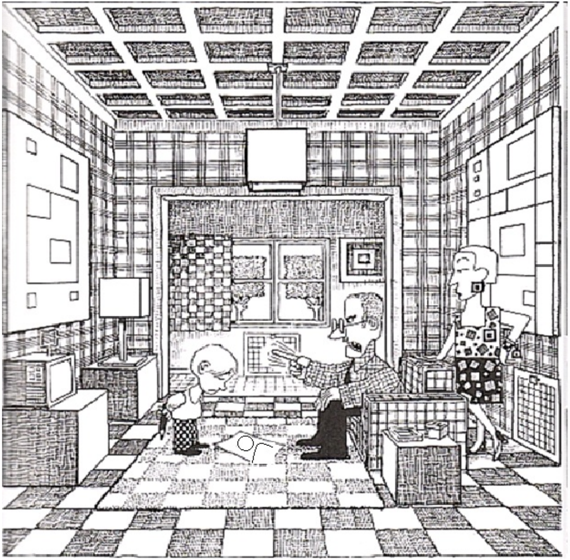
3. Elliptic curve discrete logarithm problem: Given an elliptic curve $E/\mathbb{F}_q$ and $P, Q \in E(\mathbb{F}_{q^k})$, find an integer x (if one exists) such that, $xP = Q$

Elliptic curves



borrowed from Quino.

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- Interest: smaller keys than usual cryptosystems (RSA, ElGamal, ...)

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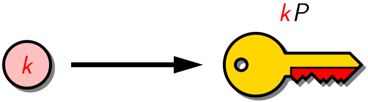

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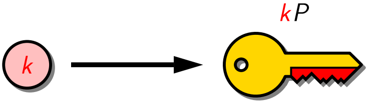

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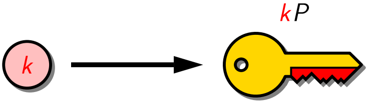

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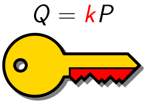

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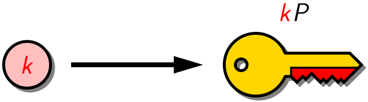

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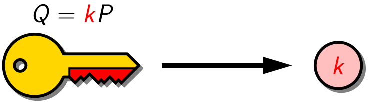

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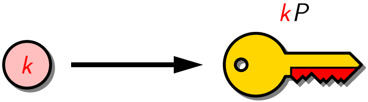

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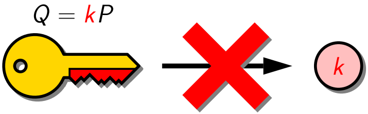

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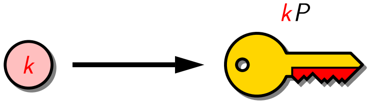

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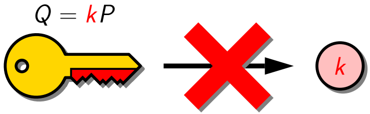

- We assume that the discrete logarithm problem (DLP) in $\mathbb{G}_1$ is hard

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- Immediate property: for any two integers k_1 and k_2

$$\hat{e}(k_1 Q, k_2 R) = \hat{e}(Q, R)^{k_1 k_2}$$

Pairing-based cryptography: The MOV attack

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  - Menezes-Okamoto-Vanstone and Frey-Rück attacks, 1993 and 1994

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- Short digital signatures
  - Boneh–Lynn–Shacham, 2001
  - Zang–Safavi-Naini–Susilo, 2004
- ...

Pairing-based cryptography: How to define pairings using elliptic curves

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  - $\mathbb{F}_q$, a finite field, with $q = 2^m$, 3^m or p
  - E, an elliptic curve defined over $\mathbb{F}_q$
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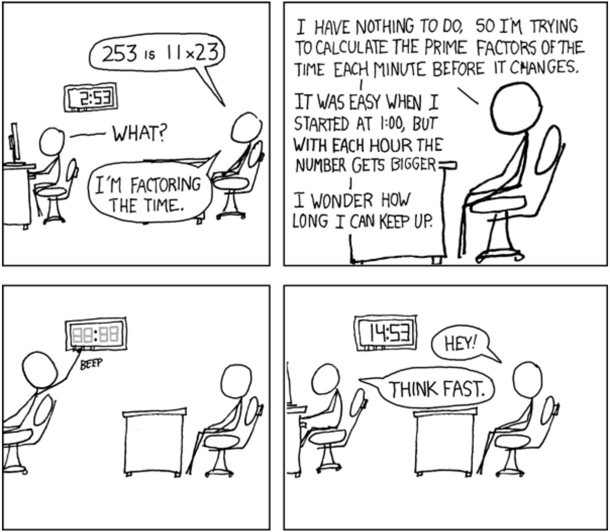
- k is the embedding degree, the smallest integer such that $\ell | q^k - 1$
  - usually large for ordinary elliptic curves
  - bounded in the case of supersingular elliptic curves
    (4 in characteristic 2; 6 in characteristic 3; and 2 in characteristic > 3)

Time complexity



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- A subexponential-time algorithm as one whose running time is of the form,

$$L_q[\alpha, c] = e^{c(\log q)^{\alpha}(\log \log q)^{1-\alpha}},$$

where $0 < \alpha < 1$, and c is a constant.
$\alpha = 0$: polynomial $\quad \alpha = 1$: fully exponential

Historic major developments

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- Elliptic curve discrete logarithm over ($\mathbb{F}_q$)
  - Pollard (1978): $q^{1/2}$.

Recommended key sizes

Security in bits	RSA $\|\|N\|\|_2$	DL: $\mathbb{F}_p$ $\|\|p\|\|_2$	DL: $\mathbb{F}_{2^m}$ m	ECC $\|\|q\|\|_2$
80	1024	1024	1500	160
112	2048	2048	3500	224
128	3072	3072	4800	256
192	7680	7680	12500	384
256	15360	15360	25000	512

Pairing-based cryptography: Believed security circa 2012 for supersingular curves

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- $\mathbb{F}_{2^m}$: simpler finite field arithmetic

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Higher security $(\sim 2^{128})$	$m = 1103$	$m = 503$	$\|p\| \approx 1536$ bits

- $\mathbb{F}_{2^m}$: simpler finite field arithmetic
- $\mathbb{F}_{3^m}$: smaller field extension

Pairing-based cryptography: Believed security circa 2012 for supersingular curves

$$\hat{e} : E(\mathbb{F}_q)[\ell] \times E(\mathbb{F}_q)[\ell] \to \mu_\ell \subseteq \mathbb{F}_{q^k}^\times$$

- The embedding degree k depends on the field characteristic q

Base field ($\mathbb{F}_q$)	$\mathbb{F}_{2^m}$	$\mathbb{F}_{2^m}$	$\mathbb{F}_p$
Embedding degree (k)	4	6	2
Lower security ($\sim 2^{64}$)	$m = 239$	$m = 97$	$\lvert p\rvert \approx 256$ bits
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Index-Calculus Algorithms for DLP in $\mathbb{F}_{q^n}$

The elements of $\mathbb{F}_{q^n}$ can be viewed as the polynomials of degree at most $n-1$ in the ring $\mathbb{F}_q[X]$.

Field arithmetic is performed by means of a degree n polynomial whose coefficients are in $\mathbb{F}_q$, irreducible over the base field $\mathbb{F}_q$.

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Index-Calculus Algorithms for DLP in $\mathbb{F}_{q^n}$ comprisses four main phases:

1. Factor base: Composed by all irreducible polynomials of degree $\leq t$
2. Relation generation: Find individual linear relations of the logarithms of factor base elements
3. Linear system: Obtain the logarithms of factor base elements by solving a linear system of equations that arises from collecting all the relations found in the previous phase
4. Descent: Compute the logarithm of the given element

Attacks on discrete log computation over small char $\mathbb{F}_{q^n}$: Main developments in the last 30+ years

Let Q be defined as $Q = q^n$.

- Hellman-Reyneri 1982: Index-calculus $L_Q[\frac{1}{2}, 1.414]$
- Coppersmith 1984: $L_Q[\frac{1}{3}, 1.526]$
- Joux-Lercier 2006: $L_Q[\frac{1}{3}, 1.442]$ when q and n are "balanced"
- Hayashi et al. 2012: Used an improved version of the Joux-Lercier method to compute discrete logs over the field $\mathbb{F}_{3^{6\cdot 97}}$
- Joux 2012: $L_Q[\frac{1}{3}, 0.961]$ when q and n are "balanced"
- Joux 2013: $L_Q[\frac{1}{4} + o(1), c]$ when $Q = q^{2m}$ and $q \approx m$
- Göloğlu et al. 2013: similar to Joux 2013, BPA @ Crypto'2013

Attacks on discrete log computation over small char $\mathbb{F}_{q^n}$: security level consequences

Let us assume that one wants to compute discrete logarithms in the field $\mathbb{F}_{q^n}$, with $q = 3^6$, $n = 509$ Notice that the multiplicative group size of that field is,

$$\#\mathbb{F}_{3^{6\cdot 509}} = \lceil \log_2(3) \cdot 6 \cdot 509 \rceil = 4841 \text{ bits.}$$

Algorithm	Time complexity	Equivalent bit security level
Hellman-Reyneri 1982	$L_Q[\frac{1}{2}, 1.414]$	337
Coppersmith 1984	$L_Q[\frac{1}{3}, 1.526]$	134
Joux-Lercier 2006	$L_Q[\frac{1}{3}, 1.442]$	126

2010: The year we make contact

[~~2010~~] 2013: The year we make contact

~~[2010]~~ 2013: The year we make contact

- Feb 11 2013 Joux: $\mathbb{F}_{2^{1778}} = \mathbb{F}_{(2^7)^{2\cdot 127}}$.
  - 215 CPU hours
- Feb 19 2013 Göloğlu et al.: $\mathbb{F}_{2^{1971}} = \mathbb{F}_{(2^9)^{3\cdot 73}}$.
  - 3,132 CPU hours
- Mar 22 2013 Joux: $\mathbb{F}_{2^{4080}} = \mathbb{F}_{(2^8)^{2\cdot 255}}$.
  - 14,100 CPU hours
- April 6 2013, Barbulescu et al.: $\mathbb{F}_{2^{809}}$,
  - notice that 809 is a prime number.
  - using conventional techniques based on the Coppersmith algorithm
  - 30,000+ CPU hours
- Apr 11 2013 Göloğlu et al.: $\mathbb{F}_{2^{6120}} = \mathbb{F}_{(2^8)^{3\cdot 255}}$.
  - 750 CPU hours
- May 21 2013 Joux: $\mathbb{F}_{2^{6168}} = \mathbb{F}_{(2^8)^{3\cdot 257}}$.
  - 550 CPU hours

A Quasi-Polynomial Time Algorithm

(June 19 2013) Barbulescu-Gaudry-Joux-Thomé

- Let q be a prime power, and let $n \leq q + 2$.
- The DLP in $\mathbb{F}_{q^{2 \cdot n}}$ can be solved in time

$$q^{O(\log n)}$$

- In the case where $n \approx q$, the DLP in $\mathbb{F}_{q^{2 \cdot n}} = \mathbb{F}_Q$ can be solved in time,

$$\log Q^{O(\log \log Q)}$$

This is smaller than $L_Q[\alpha, c]$ for any $\alpha > 0$ and $c > 0$.

Cryptographic implications

PJCrypto: Post-Joux Cryptography

1. Discrete log cryptography
2. Pairing-based cryptography
3. Elliptic curve cryptography

Discrete log cryptography

Diffie-Hellman, ElGamal, DSA, ...

- DL cryptography over $\mathbb{F}_p$ is not affected.
- DL cryptography over $\mathbb{F}_{2^m}$, m prime, might be affected.
- Note that $\mathbb{F}_{2^m}$ can be embedded in $\mathbb{F}_{2^{\ell m}}$ for any $l \geq 2$.
  - $\mathbb{F}_{2^{809}}$ can be embedded in $\mathbb{F}_{2^{10 \cdot 2 \cdot 809}}$. It is unlikely that the new algorithms will be faster in this larger field.

Pairing-based cryptography

Efficient discrete log algorithms in small char $\mathbb{F}_{q^n}$ fields have a direct negative impact on the security level that small characteristic symmetric pairings can offer:

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Efficient discrete log algorithms in small char $\mathbb{F}_{q^n}$ fields have a direct negative impact on the security level that small characteristic symmetric pairings can offer:

1. Supersingular elliptic curves over $\mathbb{F}_{2^n}$ with embedding degree $k = 4$
2. Supersingular elliptic curves over $\mathbb{F}_{3^n}$ with embedding degree $k = 6$
3. Supersingular genus-two curves over $\mathbb{F}_{2^n}$ with embedding degree $k = 12$
4. Elliptic curves over $\mathbb{F}_p$ with embedding degree $k = 2$
5. BN curves: Elliptic curves over $\mathbb{F}_p$ with embedding degree $k = 12$

Curves 1, 2 and 3 are potentially vulnerable to the new attacks.
Curves 4 and 5 are not affected by the new attacks.

Pairing-based cryptography

Example: Consider the supersingular elliptic curve, $Y^2 = X^3 - X + 1$, with $\#E(\mathbb{F}_{3^{509}}) = 7r$, and where, $r = (3^{509} - 3^{255} + 1)/7$ is an 804-bit prime.

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- E has embedding degree $k = 6$
- The ellliptic curve group $E(\mathbb{F}_{3^{509}})$ can be efficiently embedded in $\mathbb{F}_{3^{6\cdot 509}}$
- Question: Can logarithms in $\mathbb{F}_{3^{6\cdot 509}}$ be efficiently computed using the new algorithms? Or, at least significantly faster than the previously-known algorithms?

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- Question: Can logarithms in $\mathbb{F}_{3^{6 \cdot 509}}$ be efficiently computed using the new algorithms? Or, at least significantly faster than the previously-known algorithms?
- Note: $\mathbb{F}_{3^{6 \cdot 509}}$ can be embedded in $\mathbb{F}_{3^{6 \cdot 2 \cdot 509}}$

Elliptic curve cryptography

- The recent advances do not affect the security of (ordinary) elliptic curve cryptosystems.
- Example: NIST elliptic curve K-163:
  $E : Y^2 + XY = X^3 + X^2 + 1$ over $\mathbb{F}_{2^{163}}$ $E(\mathbb{F}_{2^{163}})$ can be embedded in $\mathbb{F}_{2^{163 \cdot 2 \cdot 17932535427373041941149514581590332356837787037}}^*$
  Elements in this large field are
  $5846006549323611672814741753598448348329118574062 \approx 2^{163}$ bits in length.

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  Elements in this large field are
  $5846006549323611672814741753598448348329118574062 \approx 2^{163}$ bits in length.
- the Eddington number, N_{Edd}, is the "provable" number of protons in the observable universe estimated as, $N_{Edd} = 136 \cdot 2^{256}$

A mainstream belief in the crypto community

- Several records broken in rapid succession by Joux, Göloğlu et al. and the Caramel team, the last of the series as of today: a discrete log computation over $\mathbb{F}_{2^{6128}} = \mathbb{F}_{(2^8)^{3\cdot 257}}$ Joux (May 21, 2013)

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- More than that, some distinguished researchers have expressed in blogs/chats the opinion that all these new developments may sooner or later bring fatal consequences for integer factorization, which eventually would lead to the death of RSA
- Nevertheless, none of the records mentioned above have attacked finite field extensions that have been previously proposed for performing pairing-based cryptography in small char

Our question

Our question: can the new attacks or a combination of them be effectively applied to compute discrete logs in finite field extensions of interest in pairing-based cryptography?

Discrete log descent

Computing discrete logarithms in $\mathbb{F}_{3^{6\cdot509}}$

- We present a concrete analysis of the DLP algorithm for computing discrete logarithms in $\mathbb{F}_{3^{6\cdot509}}$.

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- In fact, this field is embedded in the quadratic extension field $\mathbb{F}_{3^{12\cdot 509}}$, and it is in this latter field where the DLP algorithm is executed.
- Thus, we have $q = 3^6 = 729$, $n = 509$, and the size of the group is $N = 3^{12\cdot 509} - 1$. Note that $3^{12\cdot 509} \approx 2^{9681}$.
- We wish to find $\log_g h$, where g is a generator of $\mathbb{F}^*_{3^{12\cdot 509}}$ and $h \in \mathbb{F}^*_{3^{12\cdot 509}}$.

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- We wish to find $\log_g h$, where g is a generator of $\mathbb{F}^*_{3^{12\cdot 509}}$ and $h \in \mathbb{F}^*_{3^{12\cdot 509}}$.
- Once again, this field was selected to attack the elliptic curve discrete logarithm problem in $E(\mathbb{F}_{3^{509}})$, where E is the supersingular elliptic curve $Y^2 = X^3 - X + 1$ with $\#E(\mathbb{F}_{3^{509}}) = 7r$, and where $r = (3^{509} - 3^{255} + 1)/7$ is an 804-bit prime.

Computing discrete logarithms in $\mathbb{F}_{3^{6\cdot 509}}$: Main steps

Our attack was divided in three main steps

- Finding logarithms of linear polynomials
- Finding logarithms of irreducible quadratic polynomials
- Descent, divided into four different strategies:

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    1. Continued-fraction descent
    2. Classical descent
    3. QPA descent
    4. Gröbner bases descent

Finding logarithms of linear polynomials

- The factor base for linear polynomials $\mathcal{B}_1$ has size $3^{12} \approx 2^{19}$.
  - The cost of relation generation is approximately $2^{30} M_{q^2}$,
  - The cost of the linear algebra is approximately $2^{48} M_r$,

  where M_{q^2} and M_r stands for field multiplication in the field $\mathbb{F}_{q^2}$ and $\mathbb{F}_r$, respectively.

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  where M_{q^2} and M_r stands for field multiplication in the field $\mathbb{F}_{q^2}$ and $\mathbb{F}_r$, respectively.
- Note that relation generation can be effectively parallelized, unlike the linear algebra where parallelization on conventional computers provides relatively small benefits.

Finding logarithms of irreducible quadratic polynomials

- Let $u \in \mathbb{F}_{q^2}$, and let $Q(X) = X^2 + uX + v \in \mathbb{F}_{q^2}[X]$ be an irreducible quadratic.
  - Define $\mathcal{B}_{2,u}$ to be the set of all irreducible quadratics of the form $X^2 + uX + w$ in $\mathbb{F}_{q^2}[X]$

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  - The logarithms of all elements in $\mathcal{B}_{2,u}$ are found simultaneously using one application of QPA descent
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- For each $u \in \mathbb{F}_{3^{12}}$, the expected cost of computing logarithms of all quadratics in $\mathcal{B}_{2,u}$ is $2^{39} M_{q^2}$ for relation generation, and $2^{48} M_r$ for the linear algebra.
- This step is somewhat parallelizable on conventional computers since each set $\mathcal{B}_{2,u}$ can be handled by a different processor.

Descent: General approach

- Recall that we wish to compute $\log_g h$, where $h \in \mathbb{F}_{q^{2n}} = \mathbb{F}_{q^2}[X]/(I_X)$. We assume that $\deg h = n - 1$.

Descent: General approach

- Recall that we wish to compute $\log_g h$, where $h \in \mathbb{F}_{q^{2n}} = \mathbb{F}_{q^2}[X]/(I_X)$. We assume that $\deg h = n - 1$.
- The descent stage begins by multiplying h by a random power of g, namely, $h' = h \cdot g^i$ for some $i \in \mathbb{F}_r$.

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- The descent stage begins by multiplying h by a random power of g, namely, $h' = h \cdot g^i$ for some $i \in \mathbb{F}_r$.
- The descent algorithm gives $\log_g h'$ as a linear combination of logarithms of polynomials of degree at most two using the combination of four different strategies.

A descent into four steps

1. Continued-fraction descent: Starting from a polynomial of degree $n = 508$ gives its discrete log as a linear combination of logarithms of polynomials of degree at most $m = 30$

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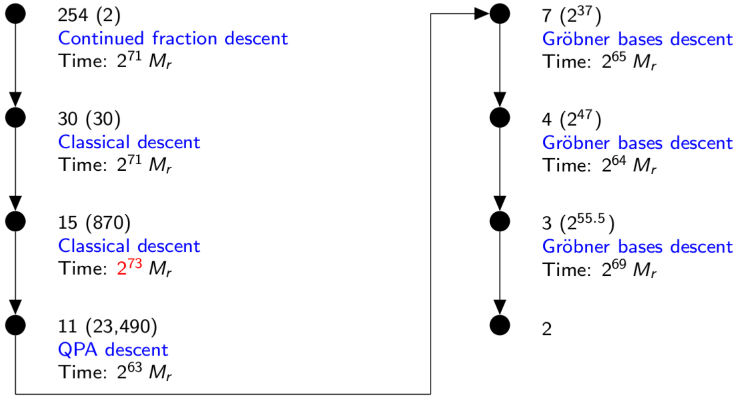
3. QPA descent: given the degree-11 polynomials of the previous step, finds their discrete log as a linear combination of logarithms of polynomials of degree at most 7
4. Gröbner bases descent: given the degree-7 polynomials of the previous step, finds their discrete log as a linear combination of logarithms of cuadratic polynomials. This concludes the descent

A positive answer: Announcing the weak field $\mathbb{F}_{3^{6\cdot 509}}$

Finding logarithms of linear polynomials	
Relation generation	$2^{22} M_r$
Linear algebra	$2^{48} M_r$
Finding logarithms of irreducible quadratic polynomials	
Relation generation	$2^{50} M_r$
Linear algebra	$2^{67} M_r$
Descent	
Continued-fraction (254 to 30)	$2^{71} M_r$
Classical (30 to 15)	$2^{71} M_r$
Classical (15 to 11)	$2^{73} M_r$
QPA (11 to 7)	$2^{63} M_r$
Gröbner bases (7 to 4)	$2^{65} M_r$
Gröbner bases (4 to 3)	$2^{64} M_r$
Gröbner bases (3 to 2)	$2^{69} M_r$

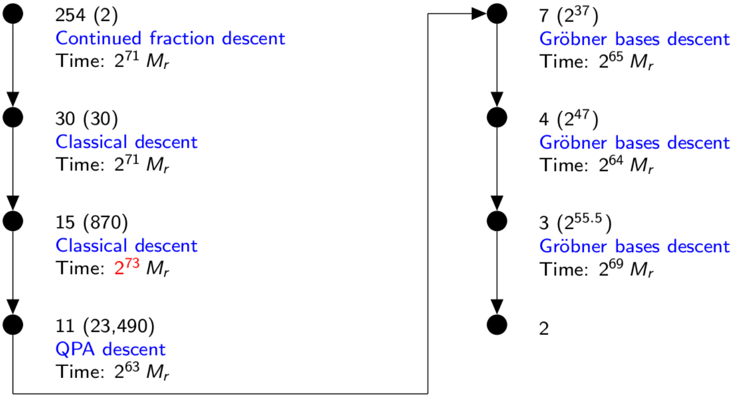
Table: Estimated costs of the main steps of the new DLP algorithm for computing discrete logarithms in $\mathbb{F}_{(3^6)^{2\cdot 509}}$. M_r denotes the costs of a multiplication modulo the 804-bit prime $r = (3^{509} - 3^{255} + 1)/7$. We also assume that 2^{22} multiplications modulo r can be performed in 1 second

Descent path for a polynomial of degree ≤ 508 over $\mathbb{F}_{3^{6\cdot 2}}$



The numbers in parentheses are the expected number of nodes at that level. 'Time' is the expected time to generate all nodes at a level.

Descent path for a polynomial of degree ≤ 508 over $\mathbb{F}_{3^{6\cdot 2}}$



The numbers in parentheses are the expected number of nodes at that level. 'Time' is the expected time to generate all nodes at a level.
All the technical details are discussed in the eprint report 2013/446

Post-Scriptum 0: Joux-Pierrot (September 9, 2013)

- Revisiting fields of pairing interest, the authors in the eprint report 2013/446, find that the running time of computing discrete logs has complexity,

$$L_Q(1/3, [(64/9) \cdot (\lambda + 1)/\lambda)]^{1/3}),$$

where λ is the degree of the polynomial that defines the field characteristic p (usually, $\lambda \leq 10$)

  - For fields of pairing interest where p is 'large' the complexity of the attack drops to,

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- The analysis is asymtoptic. In particular, this attack does not affect the 128-bit security level parameters used for the curves of class 5 in slide 21.

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- In his ECC'2013 talk, Robert Granger announced a refined version of the attack described in this presentation.

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- In his ECC'2013 talk, Robert Granger announced a refined version of the attack described in this presentation.
- This allows him to report several more weak fields in characteritic two, including, $\mathbb{F}_{2^{4\cdot 1223}}$, a field that not long ago was assumed to offer a security level of 128 bits

Merci-Thanks-Obrigado-Gracias
for your attention

borrowed from Quino.

Questions?