Lagrange Interpolation Problem

Thm Given n distinct points $(x_1, y_1) \cdots (x_n, y_n)$,

$\exists !$ polynomial interpolant of degree $\leq n-1$.

Proof of Uniqueness

Suppose $p(x)$ and $q(x)$ are polynomials of degree $\leq n-1$ that interpolate $(x_1, y_1) \cdots (x_n, y_n)$.

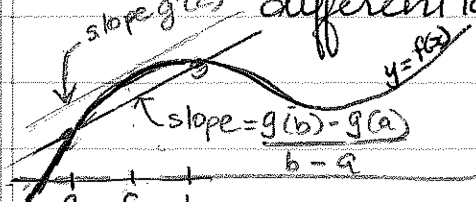
Then $p(x_i) = q(x_i)$ for $i = 1, 2, \ldots, n$.

Let $f(x) = p(x) - q(x)$

Then f is a polynomial of degree $\leq n-1$, too.

So $f(x)$ is a polynomial of degree $\leq n-1$ with $f(x_i) = 0$ $i = 1, \ldots, n$.

Mean Value Thm says that if g is continuous & differentiable on $[a,b]$, then



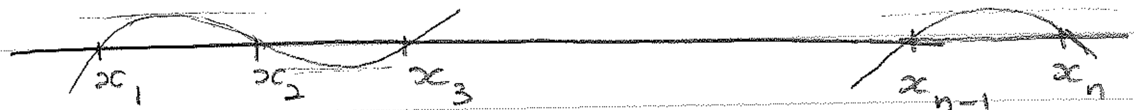
$$\frac{g(b)-g(a)}{b-a} = g'(c) \text{ for some } c \text{ betw } a \text{ \& } b.$$

Special Case of M.V. Thm

So, if $g(b) = g(a)$, then there is a c betw a and b such that
$$g'(c) = 0$$

This is actually called Rolle's Thm

$f(x_i) = 0$ for $i = 1, \ldots, n$



BY <u>MEAN VALUE THM</u> or Rolle's Thm

So, there is a point between x_1 & x_2 'cuz $f(x_1) = f(x_2) = 0$

& a point between x_2 & x_3 'cuz $f(x_2) = f(x_3) = 0$

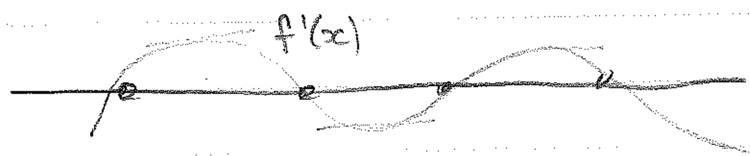
$\vdots$

& a point between x_{n-1} & x_n 'cuz $0 = f(x_{n-1}) = f(x_n)$

such that f' is 0.

(*) Now, we can say $f'(x)$ is a polynomial of degree $\leq n-2$ with at least $n-1$ points where $f' = 0$

So by MV Thm again ...
or Rolle's Thm



there are at least $n-2$ points where $f'' = 0$.

i.e

(*) $f''(x)$ is a polyn of degree $\leq n-3$ with at least $n-2$ points where $f'' = 0$

or Rolle's Thm

So by MV Thm again ...
there are at least $n-3$ points where $f''' = 0$

(2) (*) <u>i.e</u> $f'''(x)$ is a polyn of degree $\leq n-4$ with at least $(n-3)$ points where $f''' = 0$

Keep doing this...
until:
Since $f(x)$ is of degree $\leq n-1$,
then for sure
$$f^{(n-1)}(x) = \text{constant. i.e degree } 0$$

and has at least one point where
$f^{(n-1)}$ is zero.

But, if $f^{(n-1)}(x)$ is a constant that would mean that
$f^{(n-1)}$ takes the value zero everywhere

i.e $f^{(n-1)}(x) \equiv 0$

$\Rightarrow f^{(n-2)}(x) = \text{const}$ but $f^{(n-2)}(x)$ was zero somewhere so
$$f^{(n-2)}(x) \equiv 0$$
$\Rightarrow f^{(n-3)}(x) = \text{const}$ but $f^{(n-3)}(x)$ was zero somewhere so
$$f^{(n-3)}(x) \equiv 0$$
$\Rightarrow \ldots$

$\Rightarrow f(x) = \text{const}$ but $f(x)$ was zero somewhere
so $f(x) \equiv 0$

3 $\Rightarrow p(x) - q(x) \equiv 0 \Rightarrow p(x) \equiv q(x)$
$p(x)$ & $q(x)$ same function! QED