Newton's Formula for the Lagrange Interp Problem

We consider the Newton formulation (vs. Lagrange formulation)
Helpful when additional data points maybe added to the problem or when the appropriate degree of the interpolation polynomial is unknown

~ Considering degree of polynomial to find value of $f(x^*)$

given $(x_1, f(x_1))$
$(x_2, f(x_2))$
$\vdots$
$(x_n, f(x_n))$
$\vdots$

→ sometimes polynomial of too low degree i.e less data points, has estimate of $f(x^*)$ not so good

→ sometimes too high a degree polynomial i.e. using more data points gives polynomial too oscillatory so estimate not so good.

eg Given 2 points (x_1, y_1) & (x_2, y_2).

Lagrange form of polyn.

$$P(x) = y_2 \frac{x-x_1}{x_2-x_1} + y_1 \frac{x-x_2}{x_1-x_2}$$

Newton form

$$P(x) = \frac{y_2-y_1}{x_2-x_1}(x-x_1) + y_1$$

. Same polynomial – different formulation.

eg $(1,3)$ & $(2,8)$.

Lagrange $P(x) = 3\left(\frac{x-2}{1-2}\right) + 8\left(\frac{x-1}{2-1}\right)$

Newton $P(x) = \frac{5}{1}(x-1) + 3$

THM Newton Formulation of Lagrange Interpolating Polynomial

There are uniquely determined constants $c_1, c_2, \ldots, c_n$ that give the L. I. Polyn interpolating (x_1, y_1) $(x_2, y_2), \ldots, (x_n, y_n)$ in the form

$$\begin{aligned} P(x) = c_1 &+ c_2(x-x_1) + c_3(x-x_1)(x-x_2) \\ &+ c_4(x-x_1)(x-x_2)(x-x_3) \\ &+ \cdots \\ &+ c_n(x-x_1)(x-x_2)\cdots(x-x_{n-1}) \end{aligned}$$

Nested Notation

$$P(x) = c_1 + (x-x_1)\left[c_2 + (x-x_2)\left[c_3 + (x-x_3)\left[c_4 + \cdots \left[c_{n-1} + (x-x_{n-1})c_n\right]\right.\right.\right.$$

The constants are:

$$P(x_1) = y_1 \iff y_1 = c_1$$

$$P(x_2) = y_2 \iff y_2 = c_1 + (x_2 - x_1)c_2$$

$$\implies c_2 = \frac{y_2 - c_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$P(x_3) = y_3 \iff y_3 = c_1 + (x_3 - x_1)\left[c_2 + (x_3 - x_2)c_3\right]$$

$$\Longrightarrow c_3 = \frac{y_3 - c_1 - c_2(x_3 - x_1)}{(x_3 - x_1)(x_3 - x_2)}$$

$$= \cdots$$

$$= \frac{1}{x_3 - x_1}\left[\frac{y_3 - y_2}{x_3 - x_2} - \underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_{c_2}\right]$$

etc.