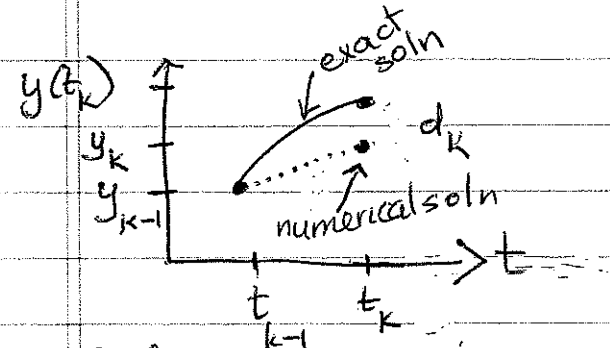
Defn The *local truncation error* is the difference bet'n the theoretical soln and the computed solution:

$$d_k = y(t_k) - y_k$$

where $y(t_k)$ is the exact soln to $\frac{dy}{dt} = f(y(t), t)$

$$y(t_{k-1}) = y_{k-1}$$



Defn

A method is said to be ORDER p if $\exists\ C \in \mathbb{R}$ such that

$$|d_k| \leq C h^{p+1}$$

i.e. $d_k = \mathcal{O}(h^{p+1})$

- Taylor Series Expansion of $g(w)$ about $w=a$

$$g(w) = g(a) + g'(a)(w-a) + g''(a)\frac{(w-a)^2}{2!} + g'''(a)\frac{(w-a)^3}{3!} + \mathcal{O}(w-a)^4$$

- Taylor Series Expansion of $g(w,r)$ about $(w,r)=(a,b)$.

See Section §5.4 in textbook

$$g(w,r) = g(a,b) + \left[\frac{\partial g}{\partial w}(a,b)\cdot(w-a) + \frac{\partial g}{\partial r}(a,b)\cdot(r-b)\right]$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 g}{\partial w^2}(a,b)(w-a)^2 + \frac{\partial^2 g}{\partial r^2}(a,b)(r-b)^2 + 2\frac{\partial g}{\partial w \partial r}(a,b)(w-a)(r-b)\right]$$
$$+ \frac{1}{3!}\left[\frac{\partial^3 g}{\partial w^3}(a,b)(w-a)^3 + 3\frac{\partial^3 g}{\partial w^2 \partial r}(a,b)(w-a)^2(r-b) + 3\frac{\partial^3 g}{\partial w \partial r^2}(a,b)(w-a)(r-b)^2 + \frac{\partial^3 g}{\partial r^3}(a,b)(r-b)^3\right]$$
$$+ \cdots$$

EULER'S METHOD

$$d_k = y(t_k) - \left[y_{k-1} + hf(y_{k-1}, t_{k-1})\right]$$

$$= \left[y(t_{k-1}) + hy'(t_{k-1}) + \frac{h^2}{2}y''(\xi)\right]$$ Taylor Expansion of $y(t)$ about t_{k-1} & eval at t_k

$$- \left[y_{k-1} + hf(y_{k-1}, t_{k-1})\right]$$

Now $y(t)$ solves $y'(t) = f(y(t), t)$ where $y(t_{k-1}) = y_{k-1}$

$$\Longrightarrow \quad y''(t) = \frac{df}{dt}(y(t), t)$$

and $y'(t_{k-1}) = f(y(t_{k-1}), t_{k-1})$

$$= f(y_{k-1}, t_{k-1})$$

So $d_k = \left[y_{k-1} + hf(y_{k-1}, t_{k-1}) + \frac{df}{dt}(\xi)\frac{h^2}{2}\right]$

$$- \left[y_{k-1} + hf(y_{k-1}, t_{k-1})\right]$$

$$= \frac{df}{dt}(\xi)\frac{h^2}{2}$$

So Euler's method has $d_k = O(h^2)$

So Euler is of order 1

MIDPOINT METHOD

$$d_k = y(t_k) - \left[y_{k-1} + hf\left(y_{k-1} + \frac{h}{2}f(y_{k-1}, t_{k-1}), t_{k-1} + \frac{h}{2}\right)\right]$$

$$= \left[y(t_{k-1}) + hy'(t_{k-1}) + \frac{h^2}{2}y''(t_{k-1}) + \frac{h^3}{6}y'''(\xi)\right]$$

$$-y_{k-1} - h\Big[f(y_{k-1}, t_{k-1}) + \frac{\partial f}{\partial y}(y_{k-1}, t_{k-1}) \cdot \frac{h}{2}f(y_{k-1}, t_{k-1})$$

$$+ \frac{\partial f}{\partial t}(y_{k-1}, t_{k-1}) \cdot \frac{h}{2}$$

$$+ \frac{\partial^2 f}{\partial y^2} \cdot \frac{1}{2} \cdot \left(\frac{h}{2}f(y_{k-1}, t_{k-1})\right)^2$$

$$+ \frac{\partial^2 f}{\partial t^2} \cdot \frac{1}{2} \cdot \left(\frac{h}{2}\right)^2$$

$$+ 2\frac{\partial f}{\partial y \partial t} \cdot \frac{1}{2}\left(\frac{h}{2}f(y_{k-1}, t_{k-1})\right) \cdot \left(\frac{h}{2}\right)$$

$$+ \mathcal{O}(h^3)\Big]$$

Now $y'(t) = f(y(t), t)$

$$\Rightarrow y''(t) = \frac{d}{dt}f(y(t), t) = \frac{\partial f}{\partial y} \cdot y'(t) + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial y} \cdot f(y(t), t) + \frac{\partial f}{\partial t}$$

So. d_k <u>Midpoint</u>

$$= y_{k-1} + hf(y_{k-1}, t_{k-1}) + \frac{h^2}{2} \cdot \left[\frac{\partial f}{\partial y} \cdot f(y(t);t) + \frac{\partial f}{\partial t}\right]\Bigg|_{\text{eval at } y_{k-1}, t_{k-1}}$$

$$+ \mathcal{O}(h^3)$$

$$- y_{k-1} - hf(y_{k-1}, t_{k-1}) - \frac{h^2}{2} \frac{\partial f}{\partial y} \cdot f \Big|_{(y_{k-1}, t_{k-1})}$$

$$- \frac{h^2}{2} \frac{\partial f}{\partial t} \Big|_{(y_{k-1}, t_{k-1})} - \text{some } h^3 \text{ stuff}$$

$$= \mathcal{O}(h^3)$$

Maybe it's higher than h^3 - like h^4 - since I was too lazy to work things out further to see if things cancelled, who knows?

Thus, midpoint method is order 2 (or maybe higher)

What is d_k for Modified Euler?

$$d_k = y(t_k) - y_{k-1} - \frac{h}{2}\Big[f(y_{k-1}, t_{k-1})$$

$$+ f(y_{k-1} + hf_{k-1}, t_{k-1} + h)\Big]$$

$$=$$

GLOBAL ERROR is the difference between the computed solution (ignoring round-off) and the true solution determined by the original initial condition at t_0.

i.e $e_N = y_N - y(t_N)$.

The global error e_N can be expressed

as a sum of N local errors coupled by factors describing the STABILITY of the equations

Roughly speaking, if the local error d_k is $\mathcal{O}(h^{p+1})$ then global error is $N \cdot \mathcal{O}(h^{p+1})$

$$= \mathcal{O}\left(\frac{1}{h}\right) \cdot \mathcal{O}(h^{p+1}) = \mathcal{O}(h^p)$$

since

$$h = \frac{t_N - t_0}{N} = \mathcal{O}\left(\frac{1}{N}\right).$$

eg In Euler's method, $p=1$ - location truncation error is $\mathcal{O}(h^2)$ - so reducing step size by ½ reduces local error by $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, but we take twice as many steps, so global error gets reduced by $2\left(\frac{1}{2}\right)^2 = \frac{1}{2}$ only. hence it is order 1.