1. Section 4.1: 2a,b; 4a,b
2. Section 4.1: 6a,b; 8a,b
3. Section 4.1: 22
4. Section 4.1: 23
5. Consider the Matlab file located at `https://www.mathworks.com/matlabcentral/fileexchange/29851-runge-kutta-4th-order-ode/content/Runge_Kutta_4.m` which implements the This implements the 4th order Runge Kutta method - see `https://en.wikipedia.org/wiki/Runge-Kutta_methods` - to solve $dy/dt = f(t, y) = 3e^{-t} - 0.4y$ with $y(1) = 5$.
   (a) Modify this routine to solve $dy/dt = f(t, y) = t^6$ with $y(0) = 0.2$ using a step size of $h = 0.1$ up to time $t = 3$.
   Plot the exact solution and the RK4 solution on the same plot. Use a legend to distinguish the two curves.
   (b) Modify this routine (and give it a new name) to solve $dy/dt = f(t, y) = t^6$ with $y(0) = 0.2$ using a step size of $h = 0.1$ up to time $t = 3$ but using Euler's method.
   Plot the exact solution and the Euler's solution on the same plot. Use a legend to distinguish the two curves.
   (c) Redo the two problems above but with $f(t, y) = t^5$.
   (d) Redo the two problems above but with $f(t, y) = t^4$.
   (e) Discuss all your results.
6. Section 5.2: 5a, 6b,d; 7a, 8b,d
7. Section 5.2: 9

is the first class of unstable methods we have encountered, and these techniques would be avoided if it were possible. However, in addition to being used for computational purposes, the formulas are needed for approximating the solutions of ordinary and partial differential equations.

EXERCISE SET 4.1

1. Use the forward-difference and backward-difference formulas to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
0.5	0.4794	
0.6	0.5646	
0.7	0.6442	

b.

x	$f(x)$	$f'(x)$
0.0	0.00000	
0.2	0.74140	
0.4	1.3718	

2. Use the forward-difference and backward-difference formulas to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
−0.3	1.9507	
−0.1	2.0421	
−0.1	2.0601	

b.

x	$f(x)$	$f'(x)$
1.0	1.0000	
1.2	1.2625	
1.4	1.6595	

3. The data in Exercise 1 were taken from the following functions. Compute the actual errors in Exercise 1, and find error bounds using the error formulas.

a. $f(x) = \sin x$ b. $f(x) = e^x - 2x^2 + 3x - 1$

4. The data in Exercise 2 were taken from the following functions. Compute the actual errors in Exercise 2, and find error bounds using the error formulas.

a. $f(x) = 2\cos 2x - x$ b. $f(x) = x^2 \ln x + 1$

5. Use the most accurate three-point formula to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	
1.4	16.44465	

b.

x	$f(x)$	$f'(x)$
8.1	16.94410	
8.3	17.56492	
8.5	18.19056	
8.7	18.82091	

c.

x	$f(x)$	$f'(x)$
2.9	−4.827866	
3.0	−4.240058	
3.1	−3.496909	
3.2	−2.596792	

d.

x	$f(x)$	$f'(x)$
2.0	3.6887983	
2.1	3.6905701	
2.2	3.6688192	
2.3	3.6245909	

6. Use the most accurate three-point formula to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
−0.3	−0.27652	
−0.2	−0.25074	
−0.1	−0.16134	
0	0	

b.

x	$f(x)$	$f'(x)$
7.4	−68.3193	
7.6	−71.6982	
7.8	−75.1576	
8.0	−78.6974	

c.

x	$f(x)$	$f'(x)$
1.1	1.52918	
1.2	1.64024	
1.3	1.70470	
1.4	1.71277	

d.

x	$f(x)$	$f'(x)$
−2.7	0.054797	
−2.5	0.11342	
−2.3	0.65536	
−2.1	0.98472	

7. The data in Exercise 5 were taken from the following functions. Compute the actual errors in Exercise 5, and find error bounds using the error formulas.
   a. $f(x) = e^{2x}$
   b. $f(x) = x \ln x$
   c. $f(x) = x \cos x - x^2 \sin x$
   d. $f(x) = 2(\ln x)^2 + 3 \sin x$
8. The data in Exercise 6 were taken from the following functions. Compute the actual errors in Exercise 6, and find error bounds using the error formulas.
   a. $f(x) = e^{2x} - \cos 2x$
   b. $f(x) = \ln(x+2) - (x+1)^2$
   c. $f(x) = x \sin x + x^2 \cos x$
   d. $f(x) = (\cos 3x)^2 - e^{2x}$
9. Use the formulas given in this section to determine, as accurately as possible, approximations for each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
2.1	−1.709847	
2.2	−1.373823	
2.3	−1.119214	
2.4	−0.9160143	
2.5	−0.7470223	
2.6	−0.6015966	

b.

x	$f(x)$	$f'(x)$
−3.0	9.367879	
−2.8	8.233241	
−2.6	7.180350	
−2.4	6.209329	
−2.2	5.320305	
−2.0	4.513417	

10. Use the formulas given in this section to determine, as accurately as possible, approximations for each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
1.05	−1.709847	
1.10	−1.373823	
1.15	−1.119214	
1.20	−0.9160143	
1.25	−0.7470223	
1.30	−0.6015966	

b.

x	$f(x)$	$f'(x)$
−3.0	16.08554	
−2.8	12.64465	
−2.6	9.863738	
−2.4	7.623176	
−2.2	5.825013	
−2.0	4.389056	

11. The data in Exercise 9 were taken from the following functions. Compute the actual errors in Exercise 9, and find error bounds using the error formulas and Maple.
    a. $f(x) = \tan x$
    b. $f(x) = e^{x/3} + x^2$
12. The data in Exercise 10 were taken from the following functions. Compute the actual errors in Exercise 10, and find error bounds using the error formulas and Maple.
    a. $f(x) = \tan 2x$
    b. $f(x) = e^{-x} - 1 + x$
13. Use the following data and the knowledge that the first five derivatives of f are bounded on $[1, 5]$ by 2, 3, 6, 12, and 23, respectively, to approximate $f'(3)$ as accurately as possible. Find a bound for the error.

x	1	2	3	4	5
$f(x)$	2.4142	2.6734	2.8974	3.0976	3.2804

14. Repeat Exercise 13, assuming instead that the third derivative of f is bounded on $[1, 5]$ by 4.
15. Repeat Exercise 1 using four-digit rounding arithmetic, and compare the errors to those in Exercise 3.
16. Repeat Exercise 5 using four-digit chopping arithmetic, and compare the errors to those in Exercise 7.

17. Repeat Exercise 9 using four-digit rounding arithmetic, and compare the errors to those in Exercise 11.

18. Consider the following table of data:

x	0.2	0.4	0.6	0.8	1.0
$f(x)$	0.9798652	0.9177710	0.808038	0.6386093	0.3843735

   a. Use all the appropriate formulas given in this section to approximate $f'(0.4)$ and $f''(0.4)$.
   b. Use all the appropriate formulas given in this section to approximate $f'(0.6)$ and $f''(0.6)$.

19. Let $f(x) = \cos \pi x$. Use Eq. (4.9) and the values of $f(x)$ at $x = 0.25, 0.5$, and 0.75 to approximate $f''(0.5)$. Compare this result to the exact value and to the approximation found in Exercise 15 of Section 3.4. Explain why this method is particularly accurate for this problem, and find a bound for the error.

20. Let $f(x) = 3xe^x - \cos x$. Use the following data and Eq. (4.9) to approximate $f''(1.3)$ with $h = 0.1$ and with $h = 0.01$.

x	1.20	1.29	1.30	1.31	1.40
$f(x)$	11.59006	13.78176	14.04276	14.30741	16.86187

   Compare your results to $f''(1.3)$.

21. Consider the following table of data:

x	0.2	0.4	0.6	0.8	1.0
$f(x)$	0.9798652	0.9177710	0.8080348	0.6386093	0.3843735

   a. Use Eq. (4.7) to approximate $f'(0.2)$.
   b. Use Eq. (4.7) to approximate $f'(1.0)$.
   c. Use Eq. (4.6) to approximate $f'(0.6)$.

22. Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ that uses $f(x_0 - h)$, $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$, and $f(x_0 + 3h)$. [*Hint:* Consider the expression $Af(x_0 - h) + Bf(x_0 + h) + Cf(x_0 + 2h) + Df(x_0 + 3h)$. Expand in fourth Taylor polynomials, and choose A, B, C, and D appropriately.]

23. Use the formula derived in Exercise 22 and the data of Exercise 21 to approximate $f'(0.4)$ and $f'(0.8)$.

24. a. Analyze the round-off errors, as in Example 4, for the formula

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f''(\xi_0).$$

   b. Find an optimal $h > 0$ for the function given in Example 2.

25. In Exercise 10 of Section 3.3, data were given describing a car traveling on a straight road. That exercise asked you to predict the position and speed of the car when $t = 10$ s. Use the following times and positions to predict the speed at each time listed.

Time	0	3	5	8	10	13
Distance	0	225	383	623	742	993

26. In a circuit with impressed voltage $\mathcal{E}(t)$ and inductance L, Kirchhoff's first law gives the relationship

$$\mathcal{E}(t) = L\frac{di}{dt} + Ri,$$

   where R is the resistance in the circuit and i is the current. Suppose we measure the current for several values of t and obtain

t	1.00	1.01	1.02	1.03	1.0
i	3.10	3.12	3.14	3.18	3.24

where t is measured in seconds, i is in amperes, the inductance L is a constant 0.98 henries, and the resistance is 0.142 ohms. Approximate the voltage $\mathcal{E}(t)$ when $t = 1.00, 1.01, 1.02, 1.03$, and 1.04.

27. All calculus students know that the derivative of a function f at x can be defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Choose your favorite function f, nonzero number x, and computer or calculator. Generate approximations $f_n'(x)$ to $f'(x)$ by

$$f_n'(x) = \frac{f(x+10^{-n}) - f(x)}{10^{-n}},$$

for $n = 1, 2, \ldots, 20$, and describe what happens.

28. Derive a method for approximating $f'''(x_0)$ whose error term is of order h^2, by expanding the function f in a fourth Taylor polynomial about x_0 and evaluating at $x_0 \pm h$ and $x_0 \pm 2h$.

29. Consider the function

$$e(h) = \frac{\varepsilon}{h} + \frac{h^2}{6}M,$$

where M is a bound for the third derivative of a function. Show that $e(h)$ has a minimum at $\sqrt[3]{3\varepsilon/M}$.

4.2 Richardson's Extrapolation

Lewis Fry Richardson (1881–1953) was the first person to systematically apply mathematics to weather prediction while working in England for the Meteorological Office. As a conscientious objector during World War I he wrote extensively about the economic futility of warfare, using systems of differential equations to model rational interactions between countries. The extrapolation technique that bears his name was the rediscovery of a technique with roots that are at least as old as Christian Huygens (1629–1695), and possibly Archimedes (287–212 BCE).

Richardson's extrapolation is used to generate high-accuracy results while using low-order formulas. Although the name attached to the method refers to a paper written by L.F. Richardson and J.A. Gaunt [RG] in 1927, the idea behind the technique is much older. An interesting article regarding the history and application of extrapolation can be found in [Joy].

Extrapolation can be applied whenever it is known that an approximation technique has an error term with a predictable form, one that depends on a parameter, usually the step size h. Suppose that for each number $h \neq 0$ we have a formula $N(h)$ that approximates an unknown value M and that the truncation error involved with the approximation has the form

$$M - N(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots,$$

for some collection of unknown constants $K_1, K_2, K_3, \ldots$.

Since the truncation error is $O(h)$, we would expect, for example, that

$$M - N(0.1) \approx 0.1K_1, \quad M - N(0.01) \approx 0.01K_1,$$

and, in general, $M - N(h) \approx K_1 h$, unless there was a large variation in magnitude among the constants $K_1, K_2, K_3, \ldots$.

The object of extrapolation is to find an easy way to combine the rather inaccurate $O(h)$ approximations in an appropriate way to produce formulas with a higher-order and $u_0, u_1, \ldots, u_N$ be the approximations obtained using Eq. (5.11). If $|\delta_i| < \delta$ for each $i = 0, 1, \ldots, N$ and the hypotheses of Theorem 5.9 hold for Eq. (5.12), then

$$|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)[e^{L(t_i-a)} - 1] + |\delta_0| e^{L(t_i-a)}, \tag{5.13}$$

for each $i = 0, 1, \ldots, N$. ∎

The error bound (5.13) is no longer linear in h. In fact, since

$$\lim_{h\to 0}\left(\frac{hM}{2} + \frac{\delta}{h}\right) = \infty,$$

the error would be expected to become large for sufficiently small values of h. Calculus can be used to determine a lower bound for the step size h. Letting $E(h) = (hM/2) + (\delta/h)$ implies that $E'(h) = (M/2) - (\delta/h^2)$.

If $h < \sqrt{\frac{2\delta}{M}}$, then $E'(h) < 0$ and $E(h)$ is decreasing.

If $h > \sqrt{\frac{2\delta}{M}}$, then $E'(h) > 0$ and $E(h)$ is increasing.

The minimal value of $E(h)$ occurs when

$$h = \sqrt{\frac{2\delta}{M}}. \tag{5.14}$$

Decreasing h beyond this value tends to increase the total error in the approximation. Normally, however, the value of δ is sufficiently small that this lower bound for h does not affect the operation of Euler's method.

EXERCISE SET 5.2

1. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0$, with $h = 0.5$
   b. $y' = 1 + (t-y)^2, \quad 2 \le t \le 3, \quad y(2) = 1$, with $h = 0.5$
   c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2$, with $h = 0.25$
   d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.25$
2. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   a. $y' = e^{t-y}, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.5$
   b. $y' = \dfrac{1+t}{1+y}, \quad 1 \le t \le 2, \quad y(1) = 2$, with $h = 0.5$
   c. $y' = -y + ty^{1/2}, \quad 2 \le t \le 3, \quad y(2) = 2$, with $h = 0.25$
   d. $y' = t^{-2}(\sin 2t - 2ty), \quad 1 \le t \le 2, \quad y(1) = 2$, with $h = 0.25$
3. The actual solutions to the initial-value problems in Exercise 1 are given here. Compare the actual error at each step to the error bound.

a. $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$
b. $y(t) = t + (1/1 - t)$
c. $y(t) = t \ln t + 2t$
d. $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$

4. The actual solutions to the initial-value problems in Exercise 2 are given here. Compare the actual error at each step to the error bound.
   a. $y(t) = \ln(e^t + e - 1)$
   b. $y(t) = \sqrt{t^2 + 2t + 6} - 1$
   c. $y(t) = \left(t - 2 + \sqrt{2}ee^{-\frac{t}{2}}\right)^2$
   d. $y(t) = \dfrac{4 + \cos 2 - \cos 2t}{2t^2}$

5. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   a. $y' = y/t - (y/t)^2, \quad 1 \le t \le 2, \quad y(1) = 1$, with $h = 0.1$
   b. $y' = 1 + y/t + (y/t)^2, \quad 1 \le t \le 3, \quad y(1) = 0$, with $h = 0.2$
   c. $y' = -(y + 1)(y + 3), \quad 0 \le t \le 2, \quad y(0) = -2$, with $h = 0.2$
   d. $y' = -5y + 5t^2 + 2t, \quad 0 \le t \le 1, \quad y(0) = \frac{1}{3}$, with $h = 0.1$

6. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   a. $y' = \dfrac{2 - 2ty}{t^2 + 1}, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.1$
   b. $y' = \dfrac{y^2}{1 + t}, \quad 1 \le t \le 2, \quad y(1) = \dfrac{-1}{\ln 2}$, with $h = 0.1$
   c. $y' = t^{-1}(y^2 + y), \quad 1 \le t \le 3, \quad y(1) = -2$, with $h = 0.2$
   d. $y' = -ty + 4ty^{-1}, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.1$

7. The actual solutions to the initial-value problems in Exercise 5 are given here. Compute the actual error in the approximations of Exercise 5.
   a. $y(t) = \dfrac{t}{1 + \ln t}$
   b. $y(t) = t\tan(\ln t)$
   c. $y(t) = -3 + \dfrac{2}{1 + e^{-2t}}$
   d. $y(t) = t^2 + \dfrac{1}{3}e^{-5t}$

8. The actual solutions to the initial-value problems in Exercise 6 are given here. Compute the actual error in the approximations of Exercise 6.
   a. $y(t) = \dfrac{2t + 1}{t^2 + 1}$
   b. $y(t) = \dfrac{-1}{\ln(t + 1)}$
   c. $y(t) = \dfrac{2t}{1 - 2t}$
   d. $y(t) = \sqrt{4 - 3e^{-t^2}}$

9. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$:

   a. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.
   b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values.
      i. $y(1.04)$
      ii. $y(1.55)$
      iii. $y(1.97)$
   c. Compute the value of h necessary for $|y(t_i) - w_i| \le 0.1$, using Eq. (5.10).

10. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2, \quad y(1) = -1,$$

with exact solution $y(t) = -1/t$: