Assignment V **Due: Wednesday 7th December, 2016**

1. Section 4.3: 1b, 3b, 5b, 7b
2. Section 4.3: 2a, 4a
3. Section 4.3: 16
4. Section 4.3: 18
5. Theorem: If f is continuously twice differentiable on $[a, b]$, then on a uniform partition of $[a, b]$

$$\left| \int_a^b f(x)\,dx - T_N \right| = \mathcal{O}(h^2).$$

   Estimate numerically the order of convergence of the following:

   (a) The Trapezoid method on a uniform grid for

$$\int_0^1 e^{-x^2}\,dx,$$

   (b) The Trapezoid method on a uniform grid for

$$\int_0^1 \sqrt{x}\,dx,$$

   Discuss your results.

6. For continuous enough f, Simpson's rule is fourth order accurate. Thus, the error in approximating $\int_a^b f(x)\,dx$ by

$$S = \frac{h}{6}\left(f(a) + 4f(c) + f(b)\right),$$

   for $c = (a+b)/2$ and $h = (b-a)/2$ is roughly $2^4 = 16$ times as large as applying the basic Simpson's rule to each of the two halves of the interval $[a, c]$ and $[c, b]$ and adding the result (i.e Composite Simpson's rule with $h = (b-a)/4$).

   (a) Use this result to derive the *extrapolated Simpson's rule*.
   This is an application of *Richardson's Extrapolation* in Section 4.2 of the textbook.
   Note that in class we made an observation on the size of the errors in the Trapezoid Rule and the Midpoint Rule to derive Simpson's rule.

   (b) Find the order of this method.

7. Section 4.4: 1f, 3f
8. Section 4.4: 13a,b,c
9. Section 4.4: 22
10. Section 4.4: 23

Table 4.8

n	0	1	2	3	4
Closed formulas		0.27768018	0.29293264	0.29291070	0.29289318
Error		0.01521303	0.00003942	0.00001748	0.00000004
Open formulas	0.30055887	0.29798754	0.29285866	0.29286923	
Error	0.00766565	0.00509432	0.00003456	0.00002399	

EXERCISE SET 4.3

1. Approximate the following integrals using the Trapezoidal rule.
   a. $\int_{0.5}^{1} x^4\,dx$
   b. $\int_{0}^{0.5} \frac{2}{x-4}\,dx$
   c. $\int_{1}^{1.5} x^2 \ln x\,dx$
   d. $\int_{0}^{1} x^2 e^{-x}\,dx$
   e. $\int_{1}^{1.6} \frac{2x}{x^2-4}\,dx$
   f. $\int_{0}^{0.35} \frac{2}{x^2-4}\,dx$
   g. $\int_{0}^{\pi/4} x \sin x\,dx$
   h. $\int_{0}^{\pi/4} e^{3x} \sin 2x\,dx$
2. Approximate the following integrals using the Trapezoidal rule.
   a. $\int_{-0.25}^{0.25} (\cos x)^2\,dx$
   b. $\int_{-0.5}^{0} x \ln(x+1)\,dx$
   c. $\int_{0.75}^{1.3} \left((\sin x)^2 - 2x \sin x + 1\right)\,dx$
   d. $\int_{e}^{e+1} \frac{1}{x \ln x}\,dx$
3. Find a bound for the error in Exercise 1 using the error formula, and compare this to the actual error.
4. Find a bound for the error in Exercise 2 using the error formula, and compare this to the actual error.
5. Repeat Exercise 1 using Simpson's rule.
6. Repeat Exercise 2 using Simpson's rule.
7. Repeat Exercise 3 using Simpson's rule and the results of Exercise 5.
8. Repeat Exercise 4 using Simpson's rule and the results of Exercise 6.
9. Repeat Exercise 1 using the Midpoint rule.
10. Repeat Exercise 2 using the Midpoint rule.
11. Repeat Exercise 3 using the Midpoint rule and the results of Exercise 9.
12. Repeat Exercise 4 using the Midpoint rule and the results of Exercise 10.
13. The Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ gives the value 4, and Simpson's rule gives the value 2. What is $f(1)$?
14. The Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ gives the value 5, and the Midpoint rule gives the value 4. What value does Simpson's rule give?
15. Find the degree of precision of the quadrature formula

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

16. Let $h = (b-a)/3$, $x_0 = a$, $x_1 = a + h$, and $x_2 = b$. Find the degree of precision of the quadrature formula

$$\int_a^b f(x)\,dx = \frac{9}{4}hf(x_1) + \frac{3}{4}hf(x_2).$$

17. The quadrature formula $\int_{-1}^{1} f(x)\,dx = c_0 f(-1) + c_1 f(0) + c_2 f(1)$ is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1, and c_2.

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18. The quadrature formula $\int_0^2 f(x)\,dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$ is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1, and c_2.
19. Find the constants c_0, c_1, and x_1 so that the quadrature formula $\int_0^1 f(x)\,dx = c_0 f(0) + c_1 f(x_1)$ has the highest possible degree of precision.
20. Find the constants x_0, x_1, and c_1 so that the quadrature formula $\int_0^1 f(x)\,dx = \frac{1}{2} f(x_0) + c_1 f(x_1)$ has the highest possible degree of precision.
21. Approximate the following integrals using formulas (4.23) through (4.30). Are the accuracies of the approximations consistent with the error formulas? Which of parts (d) and (e) give the better approximation?

    a. $\displaystyle\int_0^{0.1} \sqrt{1+x}\,dx$ b. $\displaystyle\int_0^{\pi/2} (\sin x)^2\,dx$ c. $\displaystyle\int_{1.1}^{1.5} e^x\,dx$

    d. $\displaystyle\int_1^{10} \frac{1}{x}\,dx$ e. $\displaystyle\int_1^{5.5} \frac{1}{x}\,dx + \int_{5.5}^{10} \frac{1}{x}\,dx$ f. $\displaystyle\int_0^1 x^{1/3}\,dx$
22. Given the function f at the following values,

x	1.8	2.0	2.2	2.4	2.6
$f(x)$	3.12014	4.42569	6.04241	8.03014	10.46675

    approximate $\int_{1.8}^{2.6} f(x)\,dx$ using all the appropriate quadrature formulas of this section.
23. Suppose that the data of Exercise 22 have round-off errors given by the following table.

x	1.8	2.0	2.2	2.4	2.6
Error in $f(x)$	2×10^{-6}	-2×10^{-6}	-0.9×10^{-6}	-0.9×10^{-6}	2×10^{-6}

    Calculate the errors due to round-off in Exercise 22.
24. Derive Simpson's rule with error term by using

$$\int_{x_0}^{x_2} f(x)\,dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(4)}(\xi).$$

    Find a_0, a_1, and a_2 from the fact that Simpson's rule is exact for $f(x) = x^n$ when $n = 1, 2$, and 3. Then find k by applying the integration formula with $f(x) = x^4$.
25. Prove the statement following Definition 4.1; that is, show that a quadrature formula has degree of precision n if and only if the error $E(P(x)) = 0$ for all polynomials $P(x)$ of degree $k = 0, 1, \ldots, n$, but $E(P(x)) \neq 0$ for some polynomial $P(x)$ of degree $n+1$.
26. Derive Simpson's three-eighths rule, Eq. (4.25), with error term by using Theorem 4.2.
27. Derive Eq. (4.28) with error term by using Theorem 4.3.

4.4 Composite Numerical Integration

Piecewise approximation is often effective. Recall that this was used for spline and Bézier interpolation.

The Newton–Cotes formulas are generally unsuitable for use over large integration intervals. High-degree formulas would be required, and the values of the coefficients in these formulas are difficult to obtain. Also, the Newton–Cotes formulas are based on interpolatory polynomials that use equally spaced nodes, a procedure that is inaccurate over large intervals because of the oscillatory nature of high-degree polynomials. In this section, we discuss a *piecewise* approach to numerical integration that uses the low-order Newton–Cotes formulas. These are the techniques most often applied.

This produces a series of results culminating in the final summation

$$Tot = \sum_{j=0}^{n/2} f(x_{2j}) = \sum_{j=0}^{9} f(x_{2j}) = 6.392453222.$$

We then multiply by $2h$ to finish the Composite Midpoint method:

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>Tot:=evalf(2*h*Tot);
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$$Tot := 2.008248408$$

An important property shared by all the composite integration techniques is a stability with respect to round-off error. To demonstrate this, suppose we apply the Composite Simpson's rule with n subintervals to a function f on $[a, b]$ and determine the maximum bound for the round-off error. Assume that $f(x_i)$ is approximated by $\tilde{f}(x_i)$ and that

$$f(x_i) = \tilde{f}(x_i) + e_i, \quad \text{for each } i = 0, 1, \ldots, n,$$

where e_i denotes the round-off error associated with using $\tilde{f}(x_i)$ to approximate $f(x_i)$. Then the accumulated error, $e(h)$, in the Composite Simpson's rule is

$$\begin{aligned} e(h) &= \left| \frac{h}{3} \left[e_0 + 2 \sum_{j=1}^{(n/2)-1} e_{2j} + 4 \sum_{j=1}^{n/2} e_{2j-1} + e_n \right] \right| \\ &\le \frac{h}{3} \left[|e_0| + 2 \sum_{j=1}^{(n/2)-1} |e_{2j}| + 4 \sum_{j=1}^{n/2} |e_{2j-1}| + |e_n| \right]. \end{aligned}$$

If the round-off errors are uniformly bounded by ε, then

$$e(h) \le \frac{h}{3} \left[\varepsilon + 2 \left(\frac{n}{2} - 1 \right) \varepsilon + 4 \left(\frac{n}{2} \right) \varepsilon + \varepsilon \right] = \frac{h}{3} 3n\varepsilon = nh\varepsilon.$$

But $nh = b - a$, so $e(h) \le (b - a)\varepsilon$, a bound independent of h (and n). This means that, even though we may need to divide an interval into more parts to ensure accuracy, the increased computation that is required does not increase the round-off error. This result implies that the procedure is stable as h approaches zero. Recall that this was not true of the numerical differentiation procedures considered at the beginning of this chapter.

EXERCISE SET 4.4

1. Use the Composite Trapezoidal rule with the indicated values of n to approximate the following integrals.

   a. $\int_1^2 x \ln x \, dx, \quad n = 4$ **b.** $\int_{-2}^2 x^3 e^x \, dx, \quad n = 4$ **c.** $\int_0^2 \frac{2}{x^2 + 4} \, dx, \quad n = 6$

   d. $\int_0^{\pi} x^2 \cos x \, dx, \quad n = 6$ **e.** $\int_0^2 e^{2x} \sin 3x \, dx, \quad n = 8$ **f.** $\int_1^3 \frac{x}{x^2 + 4} \, dx, \quad n = 8$

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g. $\int_3^5 \frac{1}{\sqrt{x^2-4}}\,dx, \quad n=8$

h. $\int_0^{3\pi/8} \tan x\,dx, \quad n=8$

2. Use the Composite Trapezoidal rule with the indicated values of n to approximate the following integrals.

a. $\int_{-0.5}^{0.5} \cos^2 x\,dx, \quad n=4$

b. $\int_{-0.5}^{0.5} x\ln(x+1)\,dx, \quad n=6$

c. $\int_{.75}^{1.75} (\sin^2 x - 2x\sin x + 1)\,dx, \quad n=8$

d. $\int_e^{e+2} \frac{1}{x\ln x}\,dx, \quad n=8$

3. Use the Composite Simpson's rule to approximate the integrals in Exercise 1.
4. Use the Composite Simpson's rule to approximate the integrals in Exercise 2.
5. Use the Composite Midpoint rule with $n+2$ subintervals to approximate the integrals in Exercise 1.
6. Use the Composite Midpoint rule with $n+2$ subintervals to approximate the integrals in Exercise 2.
7. Approximate $\int_0^2 x^2\ln(x^2+1)\,dx$ using $h=0.25$. Use
   a. Composite Trapezoidal rule.
   b. Composite Simpson's rule.
   c. Composite Midpoint rule.
8. Approximate $\int_0^2 x^2 e^{-x^2}\,dx$ using $h=0.25$. Use
   a. Composite Trapezoidal rule.
   b. Composite Simpson's rule.
   c. Composite Midpoint rule.
9. Suppose that $f(0)=1$, $f(0.5)=2.5$, $f(1)=2$, and $f(0.25)=f(0.75)=\alpha$. Find α if the Composite Trapezoidal rule with $n=4$ gives the value 1.75 for $\int_0^1 f(x)\,dx$.
10. The Midpoint rule for approximating $\int_{-1}^1 f(x)\,dx$ gives the value 12, the Composite Midpoint rule with $n=2$ gives 5, and Composite Simpson's rule gives 6. Use the fact that $f(-1)=f(1)$ and $f(-0.5)=f(0.5)-1$ to determine $f(-1)$, $f(-0.5)$, $f(0)$, $f(0.5)$, and $f(1)$.
11. Determine the values of n and h required to approximate $\int_0^2 e^{2x}\sin 3x\,dx$ to within 10^{-4}. Use
    a. Composite Trapezoidal rule.
    b. Composite Simpson's rule.
    c. Composite Midpoint rule.
12. Repeat Exercise 11 for the integral $\int_0^\pi x^2\cos x\,dx$.
13. Determine the values of n and h required to approximate

$$\int_0^2 \frac{1}{x+4}\,dx$$

to within 10^{-5} and compute the approximation. Use
    a. Composite Trapezoidal rule.
    b. Composite Simpson's rule.
    c. Composite Midpoint rule.
14. Repeat Exercise 13 for the integral $\int_1^2 x\ln x\,dx$.
15. Let f be defined by

$$f(x)=\begin{cases} x^3+1, & 0\le x\le 0.1,\\ 1.001+0.03(x-0.1)+0.3(x-0.1)^2+2(x-0.1)^3, & 0.1\le x\le 0.2,\\ 1.009+0.15(x-0.2)+0.9(x-0.2)^2+2(x-0.2)^3, & 0.2\le x\le 0.3.\end{cases}$$

    a. Investigate the continuity of the derivatives of f.
    b. Use the Composite Trapezoidal rule with $n=6$ to approximate $\int_0^{0.3} f(x)\,dx$, and estimate error using the error bound.

c. Use the Composite Simpson's rule with $n = 6$ to approximate $\int_0^{0.3} f(x)\,dx$. Are the results more accurate than in part (b)?

16. Show that the error $E(f)$ for Composite Simpson's rule can be approximated by

$$-\frac{h^4}{180}[f'''(b) - f'''(a)].$$

[*Hint:* $\sum_{j=1}^{n/2} f^{(4)}(\xi_j)(2h)$ is a Riemann sum for $\int_a^b f^{(4)}(x)\,dx$.]

17. **a.** Derive an estimate for $E(f)$ in the Composite Trapezoidal rule using the method in Exercise 16.

b. Repeat part (a) for the Composite Midpoint rule.

18. Use the error estimates of Exercises 16 and 17 to estimate the errors in Exercise 12.

19. Use the error estimates of Exercises 16 and 17 to estimate the errors in Exercise 14.

20. In multivariable calculus and in statistics courses it is shown that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)(x/\sigma)^2}\,dx = 1,$$

for any positive σ. The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)(x/\sigma)^2}$$

is the *normal density function* with *mean* $\mu = 0$ and *standard deviation* σ. The probability that a randomly chosen value described by this distribution lies in $[a, b]$ is given by $\int_a^b f(x)\,dx$. Approximate to within 10^{-5} the probability that a randomly chosen value described by this distribution will lie in

a. $[-\sigma, \sigma]$ **b.** $[-2\sigma, 2\sigma]$ **c.** $[-3\sigma, 3\sigma]$

21. Determine to within 10^{-6} the length of the graph of the ellipse with equation $4x^2 + 9y^2 = 36$.

22. A car laps a race track in 84 seconds. The speed of the car at each 6-second interval is determined using a radar gun and is given from the beginning of the lap, in feet/second, by the entries in the following table.

Time	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84
Speed	124	134	148	156	147	133	121	109	99	85	78	89	104	116	123

How long is the track?

23. A particle of mass m moving through a fluid is subjected to a viscous resistance R, which is a function of the velocity v. The relationship between the resistance R, velocity v, and time t is given by the equation

$$t = \int_{v(t_0)}^{v(t)} \frac{m}{R(u)}\,du.$$

Suppose that $R(v) = -v\sqrt{v}$ for a particular fluid, where R is in newtons and v is in meters/second. If $m = 10$ kg and $v(0) = 10$ m/s, approximate the time required for the particle to slow to $v = 5$ m/s.

24. To simulate the thermal characteristics of disk brakes (see the following figure), D.A. Secrist and R.W. Hornbeck [SH] needed to approximate numerically the "area averaged lining temperature," T, of the brake pad from the equation

$$T = \frac{\int_{r_e}^{r_0} T(r) r\theta_p\,dr}{\int_{r_e}^{r_0} r\theta_p\,dr},$$