1. In class, after we went through some algebraic manipulations, we found that finding the cubic spline interpolant with nodes $x_0, x_1, \cdots, x_n$ and with S'' values given at the boundaries x_0, x_n reduces to solving a system of $n-1$ linear equations $Ax = b$ for $x = (c_1, \cdots, c_{n-1})$. Here,

$$A = \begin{pmatrix} 2(h_0+h_1) & h_1 & 0 & \cdots\cdots & 0 & 0 & 0 \\ h_1 & 2(h_1+h_2) & h_2 & \cdots\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots\cdots & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\ 0 & 0 & 0 & \cdots\cdots & 0 & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{pmatrix},$$

$$b = \begin{pmatrix} \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) - \frac{h_0}{2}S''(x_0) \\ \frac{3}{h_2}(a_3-a_2) - \frac{3}{h_1}(a_2-a_1) \\ \vdots \\ \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) - \frac{3}{h_{n-3}}(a_{n-2}-a_{n-3}) \\ \frac{3}{h_{n-1}}(a_n-a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) - \frac{h_{n-1}}{2}S''(x_n) \end{pmatrix}.$$

The boundary conditions are absorbed into the first and last equations. This could have also been expressed as a linear system of $n+1$ equations where $x = (c_0, \cdots, c_n)$, and

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots\cdots & 0 & 0 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \cdots\cdots & 0 & 0 & 0 & 0 \\ 0 & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} S''(x_0)/2 \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ \frac{3}{h_2}(a_3-a_2) - \frac{3}{h_1}(a_2-a_1) \\ \vdots \\ \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) - \frac{3}{h_{n-3}}(a_{n-2}-a_{n-3}) \\ \frac{3}{h_{n-1}}(a_n-a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \\ S''(x_n)/2 \end{pmatrix}$$

The first and last equations express the boundary conditions explicitly.

The code `splinetx.m`, however, uses the "not-a-knot" condition. (At least that's what it says in the prelude to the code.) The *not-a-knot* spline is used when no endpoint derivative information is available. The idea is to ensure third derivative continuity at x_1 and at x_{n-1}. Thus, one would replaces the second derivative conditions at x_0 and x_n that gave the matrices above, with those new conditions.

(a) Write down *all* the analytic equations that govern the "not-a-knot" spline. Then write down the resulting set of linear equations that must be solved. (No need to derive those that we have already derived in class; derive the new equations.)

(b) What is the matrix system $Ax = b$ that must be solved for this type of spline. Is this what the code `splinetx` does? This amounts to explaining carefully what is done in the code provided.

2. Consider the data in Table 1. Make modifications to the matrix A and vector b in `splinetx.m` to produce 2 new routines which implement the following boundary condition cases:

   (a) $S''(x_0) = \alpha \qquad S''(x_n) = \beta$

   (b) $S''(x_0) = S''(x_1) \qquad S''(x_{n-1}) = S''(x_n)$

   Discuss.
   Plot the resulting spline functions from a) using $\alpha = \beta = 0$, and b) that interpolate the data in Table 1 and describe their differences. Are the differences as you would have predicted? Explain.

x	1	3	5	7	11	13	17	21
y	1	1.3	1.5	2.3	1.9	1.5	1.5	1.9

Table 1: Table for question 2

3. Use cubic spline interpolation with free ends to find the population of the U.S. in 1965, 1975 and 2000 from the data in the Table 2.

Year	1930	1940	1950	1960	1970	1980
Pop.	123,203	131,669	150,697	179,323	203,212	226,505

Table 2: Table for question 3 – Population in thousands

   Compare the results with the results using polynomial interpolation.

4. Consider a cubic spline interpolating $f(x)$ at given values of x. As boundary conditions, take $S''(x_0)$ to be the linear extrapolant of S'' at x_1 and x_2, and take $S''(x_n)$ to be the linear extrapolant of S'' at x_{n-2} and x_{n-1}.
   Prove that this condition gives cubic spline curves that match to $f(x)$ exactly when $f(x)$ is itself a cubic.