CONTINUOUS LEAST SQUARES

Consider the following problem:

eg Approximate $\frac{1}{x}$ on $[1,3]$ by a quadratic, polynomial

Will do this in a least squares sense.

Find a_0, a_1, a_2 that minimizes $E = \int_1^3 \left[\frac{1}{x} - (a_0 + a_1 x + a_2 x^2)\right]^2 dx$

The general polynomial problem for $f(x)$ over $[a,b]$ is $E = \int_a^b \left[f(x) - \sum_{i=0}^{n} a_i x^i\right]^2 dx$

$$\frac{\partial E}{\partial a_0} = \int_1^3 2\left[\frac{1}{x} - (a_0 + a_1 x + a_2 x^2)\right] \cdot (-1)\, dx$$

$$\frac{\partial E}{\partial a_1} = \int_1^3 2\left[\frac{1}{x} - (a_0 + a_1 x + a_2 x^2)\right] \cdot (-x)\, dx$$

$$\frac{\partial E}{\partial a_2} = \int_1^3 2\left[\frac{1}{x} - (a_0 + a_1 x + a_2 x^2)\right] \cdot (-x^2)\, dx$$

For the general polynomial problem approx $f(x)$ by polynomial of degree n.

$$\frac{\partial E}{\partial a_k} = 2\int_a^b \left[f(x) - \sum_{i=0}^{n} a_i x^i\right] \cdot (-x^k)\, dx$$

$$= -2\int_a^b \left[f(x)x^k - \sum_{i=0}^{n} a_i x^{k+i}\right] dx$$

$$\frac{\partial E}{\partial a_0} = 0 \Rightarrow \int_1^3 \frac{1}{x}\,dx = \int_1^3 (a_0 + a_1 x + a_2 x^2)\,dx$$

$$= a_0 \int_1^3 dx + a_1 \int_1^3 x\,dx + a_2 \int_1^3 x^2\,dx$$

i.e. $\boxed{\ln 3 = 2a_0 + 4a_1 + \frac{26}{3}a_2}$

$$\frac{\partial E}{\partial a_1} = 0 \Rightarrow \int_1^3 1\,dx = a_0 \int_1^3 x + a_1 \int_1^3 x^2 + a_2 \int_1^3 x^3\,dx$$

$$\Rightarrow \boxed{2 = 4a_1 + \frac{26}{3}a_2 + 20a_3}$$

$$\frac{\partial E}{\partial a_2} = 0 \Rightarrow \int_1^3 x\,dx = a_0 \int_1^3 x^2\,dx + a_1 \int x^3\,dx + a_2 \int x^4\,dx$$

$$\Rightarrow \boxed{4 = \frac{26}{3}a_0 + 20a_1 + \frac{242}{5}a_2}$$

For the more general problem,

$$\frac{\partial E}{\partial a_k} = 0 \Rightarrow \int_a^b x^k f(x)\,dx = \sum_{i=0}^{n} a_i \int_a^b x^{k+i}\,dx$$

i.e. $$\int_a^b x^k f(x)\,dx = a_0 \cdot \int_a^b x^k\,dx + a_1 \int_a^b x^{k+1}\,dx + \cdots + a_n \int_a^b x^{k+n}\,dx$$

for $k = 0, 1, \dots, n$.

Continuous LSQ 2

Continuing the approx. to $\frac{1}{x}$ on $[1,3]$ by a quadratic:

NORMAL EQUATIONS

$$\begin{pmatrix} 2 & 4 & 26/3 \\ 4 & 26/3 & 20 \\ 26/3 & 20 & \frac{242}{5} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \ln 3 \\ 2 \\ 4 \end{pmatrix}$$

$$A x = b$$

A is the NORMAL MATRIX — it is square.

In general, the entries of A look like

$$\int_a^b x^{i+j}\, dx \quad i=0,1,\cdots,n \quad j=0,1,\cdots,n$$

A is square, symmetric $(n+1)\times(n+1)$ matrix

Hilbert Matrix

ILL-CONDITIONED.

Roundoff errors are magnified when solving $Ax=b$.

In example $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{matrix} 1.72353 \\ -0.93135 \\ 0.15888 \end{matrix}$

$\frac{1}{x}$ is approx by $1.72353 - 0.93135x + 0.15888x^2$ on $[1,3]$.

To approximate $\frac{1}{x}$ by a linear function on $\frac{1}{x}$

solve $\begin{pmatrix} 2 & 4 \\ 4 & \frac{26}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \ln 3 \\ 2 \end{pmatrix}$

$\frac{1}{x}$ is approx. by $1.14098 - 0.29584x$ on $[1,3]$.

<u>Note</u>: solving new matrix takes as long as before
Previous soln does not help.

Let $\{_i\ \varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)\}$ be a BASIS

for polynomials of degree n or whatever functionspace you wa

eg $\varphi_0(x)=1$, $\varphi_1(x)=x, \ldots, \varphi_n(x) = x^n$

eg $\varphi_0(x)=1$, $\varphi_1(x)=\cos x, \ldots, \varphi_n(x)=\cos nx$

so that $p(x)$ is UNIQUELY determined by a

LINEAR COMBINATION of the $\varphi_i(x)$'s

$$p(x) = \underline{a}_0\varphi_0(x) + \underline{a}_1\varphi_1(x) + \cdots + \underline{a}_n\varphi_n(x).$$

where $p(x)$ is in the target function space

Approximate $f(x)$ on $[a,b]$ by a function in the function space.

Minimize $E = \int_a^b \left[f(x) - \sum_{i=0}^{n} a_i\varphi_i(x)\right]^2 dx$

$$\frac{\partial E}{\partial a_k} = -2\int_a^b \left[f(x) - \sum_{i=0}^{n} a_i\varphi_i(x)\right]\cdot\varphi_k(x)\,dx$$

$$= -2\int_a^b f(x)\varphi_k(x)\,dx + 2\sum_{i=0}^{n} a_i\int \varphi_i(x)\varphi_k(x)\,dx$$

Set $\frac{\partial E}{\partial a_k}=0$ for $k=0,1,\ldots,n$. gives the linear system

$$\int_a^b f(x)\varphi_k(x)\,dx = a_0\int_a^b \varphi_0(x)\varphi_k(x)\,dx + a_1\int_a^b \varphi_1(x)\varphi_k(x)\,dx$$

$$+ \cdots + a_n\int_a^b \varphi_n(x)\varphi_k(x)\,dx$$

for $k=0,1,\ldots,n$

Continuous

LSQ 5

$$\begin{bmatrix} & & \\ & \int_a^b \varphi_j(x)\varphi_k(x)\,dx & \\ & & \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_k \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \int_a^b f(x)\varphi_0(x)\,dx \\ \vdots \\ \int_a^b f(x)\varphi_k(x)\,dx \\ \vdots \\ \int_a^b f(x)\varphi_n(x)\,dx \end{bmatrix}$$

<u>Now</u>, suppose we choose the basis $\{\varphi_0(x), \dots, \varphi_n(x)\}$ so that it is orthogonal on $[a,b]$

i.e. $$\int_a^b \varphi_k(x)\varphi_r(x)\,dx = \begin{cases} 0 & \text{if } r \neq k \\ \tau_k & \text{if } r = k \end{cases}$$

Thus, normal matrix is

$$\begin{bmatrix} \tau_0 & 0 & \cdots & & \\ 0 & \tau_1 & \cdots & 0 & \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & & \tau_k & 0 \\ & & & & \ddots \\ & & & & \tau_n \end{bmatrix}$$ diagonal!

so $$a_k = \frac{\int_a^b f(x)\varphi_k(x)\,dx}{\tau_k}.$$

Continuous LSQ (6)

Taking out or adding basis elements does not <u>undo</u> results of previous soln.