BOUNDARY CONDITIONS for Cubic Spline

(a) NATURAL SPLINE $S''(x_0) = 0$ & $S''(x_n) = 0$

Makes end cubes approach linearity

(drafting device does precisely this)

$S''(x_0) = 0 \Rightarrow c_0 = 0$ Recall $c_k = \frac{1}{2} S_k''(x_k)$

$S''(x_n) = 0 \Rightarrow c_n = 0$

Generalize to $S''(x_0) = \alpha$ and $S''(x_N) = \beta$

So strike out first & last columns of matrix since c_0 & c_n known to be zero.

See HW3, question 1

$\underset{(n-1)\times(n-1)}{A}\, x = b$

A is an $(n-1)\times(n-1)$ tridiagonal matrix where kth row has

$\cdots\; h_{k-1} \quad 2(h_{k-1}+h_k) \quad h_k \cdots$

(b) $\begin{cases} S'(x_0) = A \text{ given} \\ S'(x_n) = B \text{ given} \end{cases}$ slopes at specified values

$\Rightarrow b_0 + 2c_0(x-x_0) + 3d_0(x-x_0)^2 \Big|_{x_0} = A$

$\Rightarrow b_0 = A \Rightarrow \dfrac{a_1 - a_0}{h_0} - (c_1 + 2c_0)\dfrac{h_0}{3} = A$

$\Rightarrow 2h_0 c_0 + h_0 c_1 = 3\left(\dfrac{y_1 - y_0}{h_0} - A\right)$

& "whatever"

similarly for $S'(x_n)$

(c) $\begin{cases} S''(x_0) = S''(x_1) \\ S''(x_n) = S''(x_{n-1}) \end{cases}$ end cubics approach parabolas at their extremities

$\Rightarrow c_0 = c_1$ because $d_0 = 0$

$c_n = c_{n-1}$ $\quad d_{N-1} = 0$

So don't need to put c_0 & c_n in the list of unknowns

Replace the eq^n

$$h_0 c_0 + 2(h_0+h_1)c_1 + h_1 c_2 = 3\left[\frac{y_2-y_1}{h_1} - \frac{y_1-y_0}{h_0}\right]$$

with $\quad (3h_0+2h_1)c_1 + h_1 c_2 =$ "

and, Replace the eq^n.

$$h_{n-2}c_{n-2} + 2(h_{n-2}+h_{n-1})c_{n-1} + c_n h_{n-1} = 3\left[\frac{y_n-y_{n-1}}{h_{n-1}} - \frac{y_{n-1}-y_{n-2}}{h_{n-2}}\right]$$

with $\quad h_{n-2}c_{n-2} + (2h_{n-2}+3h_{n-1})c_{n-1} =$

$$3\left[\frac{y_n-y_{n-1}}{h_{n-1}} - \frac{y_{n-1}-y_{n-2}}{h_{n-2}}\right]$$

$$\begin{bmatrix} 3h_0+2h_1 & h_1 c_2 & 0 & \cdots & 0 \\ h_1 & 2(h_1+h_2) & h_2 & \cdots & 0 \\ 0 & h_2 & 2(h_2+h_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 2h_{n-2}+3h_{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

$= \vec{b}$ (w/ 1st & last entries removed)

SPLINE

(8)

(d) Take $S''(x_0)$ as a linear extrapolation from $S''(x_1)$ & $S''(x_2)$ and
take $S''(x_n)$ as a linear extrapolation from $S''(x_{n-1})$ & $S''(x_{n-2})$.

Only this condition gives cubic spline curves that match exactly to $f(x)$ when $f(x)$ is itself a cubic. WHY?

Since $S''(x_k) = 2c_k$, this is equiv to saying
c_0 is a linear extrapolation of c_1 & c_2.
and c_n is a " " " c_{n-1} & c_{n-2}.

$$\frac{c_0 - c_1}{h_0} = \frac{c_1 - c_2}{h_1} \Rightarrow c_0 = c_1 + \frac{h_0}{h_1}(c_1 - c_2)$$

$$= \frac{h_1 + h_0}{h_1} c_1 - \frac{h_0}{h_1} c_2$$

$$\frac{c_n - c_{n-1}}{h_{n-1}} = \frac{c_{n-1} - c_{n-2}}{h_{n-2}} \Rightarrow c_n = c_{n-1} + \frac{h_{n-1}}{h_{n-2}}\left(c_{n-1} - c_{n-2}\right)$$

$$= \frac{h_{n-2} + h_{n-1}}{h_{n-2}} c_{n-1} - \frac{h_{n-1}}{h_{n-2}} c_{n-2}$$

Now alter the first & last eq^{n}s.
Solve $A\vec{x} = \vec{b}$ for $\vec{x} = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}$, then get c_0 & c_n.

SPLINE

(9) Compute a_k, b_k, d_k & get splines