Consider the following LP given in standard form:

max $z = 5x_1 + 2x_2$

Subject to

$$\begin{aligned} 3x_1 + 2x_2 &\leq 2400 \\ x_2 &\leq 800 \\ 2x_1 \quad\quad &\leq 1200 \end{aligned}$$

$$x_1, x_2 >= 0$$

The **augmented form of this LP** is the following linear system of equations:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 \quad\quad\quad &= 2400 \\ x_2 \quad + x_4 \quad &= 800 \\ 2x_1 \quad\quad\quad + x_5 &= 1200 \end{aligned}$$

along with

$$z - 5x_1 - 2x_2 = 0$$

REMARK: The constraints give a system of 3 equations but with 5 unknowns.
If we give particular values to any 2 of the unknowns x_1, x_2, x_3, x_4, x_5, then (generally speaking), we will then need to solve a system of 3 equations in now exactly 3 unknowns and we can find that single solution (of, course there are exceptions, but generally speaking).

Initialization of the Simplex Method

When all the $b_i \geq 0$, pick the original variables to be the first set of non-basic variables and the slack variables to be the basic variables to get the starting basic feasible solution.

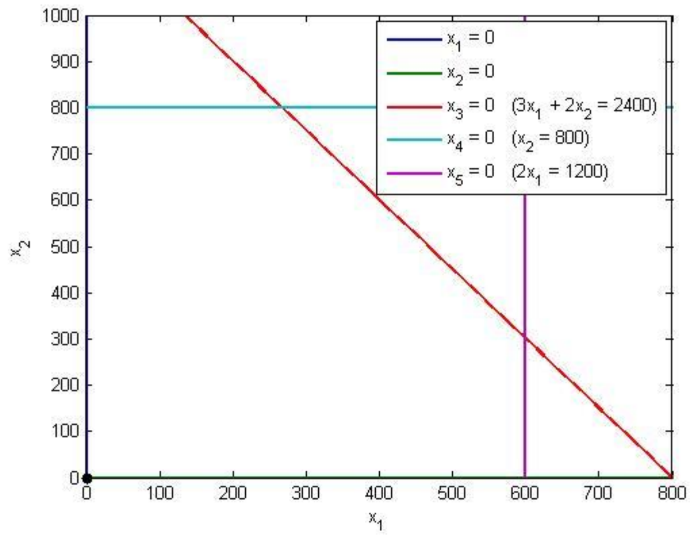
TABLEAU 1

A **tableau** shows the coefficients in the current system of equations that represents the LP. This is just the augmented matrix of the system above also noting which variables will be the basic variables that we must solve for.

Basic Variables	Eqn no.	z	x_1	x_2	x_3	x_4	x_5	RHS
z	(0)	1	-5	-2	0	0	0	0
x_3	(1)	0	3	2	1	0	0	2400
x_4	(2)	0	0	1	0	1	0	800
x_5	(3)	0	2	0	0	0	1	1200

This tableau indicates that the **non-basic variables x_1 and x_2 are set to zero**. We find the corresponding value of the basic variables by solving the system of equations when x_1 = x_2 = 0.

Therefore, the basic feasible solution associated with this tableau is **(x_1, x_2, x_3, x_4, x_5) = (0, 0, 2400, 800, 1200)** and the corresponding value of z is 0. This is associated with the CPF (x_1, x_2) = (0,0).



Optimality Test: The equation z – 5 x_1 – 2 x_2 = 0 or

z = 5 x_1 + 2 x_2 indicates that:

- if you were to increase x_1 above zero and keep x_2 fixed at 0, then z would increase;
- if you were to increase x_2 above zero and keep x_1 fixed at 0, then z would increase.

We will get the greatest increase rate in z if we increase x_1 since the coefficient magnitude of x_1 is higher than that of x_2. So, we will make x_1 basic and move it beyond 0 but keep x_2 non-basic (x_2 still equal to 0).

Geometrically speaking, we slide along the edge x_2=0 by increasing x_1 and moving away from the vertex (0,0). We have **determined our direction of movement** – the edge along which we will slide.

We can increase x_1 while still holding x_2 non-basic 0 but change the values of the other variables so that we solve the linear system of equations -- only until we are about to violate the non-negativity of a variable, i.e. until we make a new variable equal to 0. That new variable would now be non-basic and set equal to zero and x_1 would be basic.

Geometrically speaking , we increase x_1 until we hit another equality/constraint boundary. Remember, each constraint boundary corresponds to a variable (original or slack) being equal to zero.

Therefore, holding x_2 = 0,

must have 3x_1 + x_3 = 2400 and x_3 $\geq$ 0, so we can send x_1 no higher than 2400/3 = 800;

must have x_4 = 800 - this puts no constraint on x_1;

must have 2x_1 + x_5 = 1200 and still x_5 $\geq$0, so we can send x_1 no higher than 1200/2 = 600.

What is important is the value of the RHS/(positive coefficient of x_1) which we call Ratio in the tableau.

Basic Variables	Eqn no.	z	x_1	x_2	x_3	x_4	x_5	RHS	Ratio
z	(0)	1	-5	-2	0	0	0	0	
x_3	(1)	0	3	2	1	0	0	2400	2400/3
x_4	(2)	0	0	1	0	1	0	800	
x_5	(3)	0	**2**	0	0	0	1	1200	1200/2

We **pick the smallest of these ratios** as we can increase x_1 to 600 without violating any constraints. *Geometrically speaking*, we have **determined where to stop on our edge.**

So we trade in x_1 for x_5 in our basis. That is to say, we make x_1 basic and the variable that becomes zero and non-basic is x_5; x_1 enters the basis and x_5 leaves the basic as x_5 gets pushed to zero.

We solve for a new Basic Feasible Solution by constructing a new tableau in proper form using G*aussian elimination*:

i. Divide the pivot row by the pivot entry;
ii. Clear to zero all the other pivot column entries by adding/subtracting multiples of the pivot row.

The **pivot row** is the row in the tableau associated with the leaving variable (which is x_5). The **pivot column** is the column associated with the entering variable (which is x_1). The **pivot** is the coefficient of the entering variable in the pivot row.

Basic Variables	Eqn no.	z	x_1	x_2	x_3	x_4	x_5	RHS
z	(0)	1	-5	-2	0	0	0	0
x_3	(1)	0	3	2	1	0	0	2400
x_4	(2)	0	0	1	0	1	0	800
x_5	(3)	0	**2**	0	0	0	1	1200

Basic Variables	Eqn no.	z	x_1	x_2	x_3	x_4	x_5	RHS
z	(0)	1	-5	-2	0	0	0	0
x_3	(1)	0	3	2	1	0	0	2400
x_4	(2)	0	0	1	0	1	0	800
x_1	(3)	0	**1**	0	0	0	1/2	600

Eqn (0) gets replaced by Eqn(0) + 5*Eqn(3);
Eqn (1) gets replaced by Eqn(1) – 3*Eqn(3);
Eqn (2) stays the same as there is already a zero in the pivot column.

We arrive at a new tableau to describe the original LP.

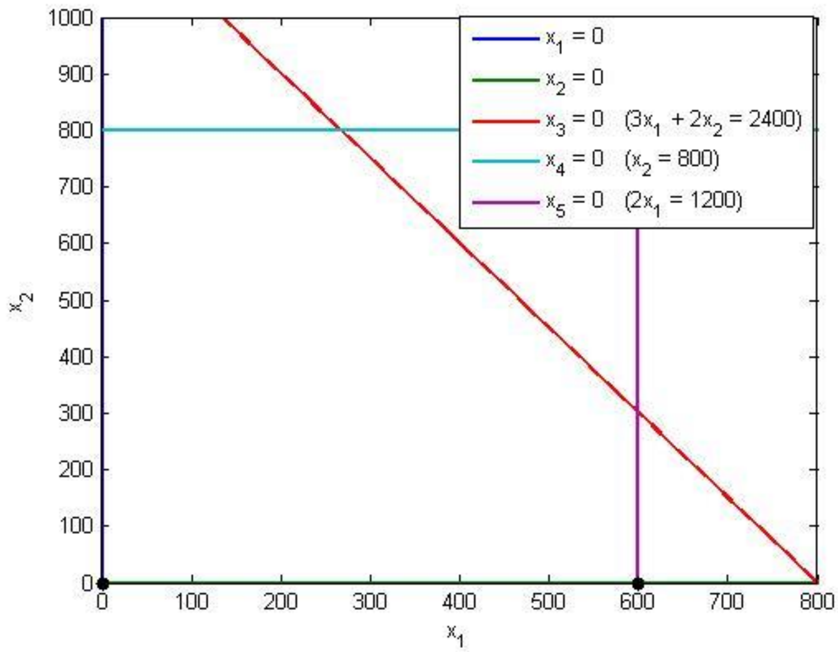
TABLEAU 2:

Basic Variables	Eqn no.	z	x_1	x_2	x_3	x_4	x_5	RHS
z	(0)	1	0	-2	0	0	5/2	3000
x_3	(1)	0	0	2	1	0	-3/2	600
x_4	(2)	0	0	1	0	1	0	800
x_1	(3)	0	1	0	0	0	1/2	600

Setting the non-basic variables x_2 and x_5 to zero and solving the system of equations for the basic variables gives x_3 = 600, x_4 = 800, and x_1 = 600.

Therefore the basic solution associated with this tableau is

(x_1, x_2, x_3, x_4, x_5) = (600, 0, 600, 800, 0) and the corresponding value of z is 3000.



Optimality Test: The equation z - 2 x_2 + 5/2 x_5 = 3000 or z = 3000 - 2 x_2 - 5/2 x_5 indicates that:

- if you were to increase x_2 above zero and keep x_5 fixed at 0, then z would increase.

So, we will make x_2 basic and move it beyond 0.
Geometrically speaking, we slide along the edge x_5=0 by increasing x_2 and moving away from the CPF(600,0). We have **determined our direction of movement** – the edge along which we will slide.

We can increase x_2 -- keeping x_5 non-basic equal to 0 but change the values of the other variables so that we solve the linear system of equations -- only until we are about to violate the non-negativity of a variable, i.e. until we make a new variable equal to 0. That new variable would now be non-basic and set equal to zero and x_2 would be basic.
Geometrically speaking , we increase x_2 until we hit another equality/constraint boundary. Remember, each constraint boundary corresponds to a variable (original or slack) being equal to zero.
Therefore, holding x_5= 0,

we must have 2x_2 + x_3 = 600 and x_3 $\geq$ 0, so we can send x_2 no higher than 600/2;
we must have x_2 + x_4 = 800 and x_4 $\geq$ 0, so we can send x_2 no higher than 800/1;
we must have x_1 = 600 and this puts no constraint on x_2.

We **pick the smallest of these ratios** as we can increase x_2 to 300 without violating any constraints.
Geometrically speaking, we have **determined where to stop on our edge.**

So we trade in x_2 for x_3 in our basis. That is to say, we make x_2 basic and the variable that becomes non-basic is x_3; x_2 enters the basis and x_3 leaves it as x_3 gets pushed to 0.

Geometrically speaking, we have **determined where to stop on our edge.**

Basic Variables	Eqn no.	z	x_1	x_2	x_3	x_4	x_5	RHS	Ratios
z	(0)	1	0	-2	0	0	5/2	3000	
x_3	(1)	0	0	2	1	0	-3/2	600	600/2
x_4	(2)	0	0	1	0	1	0	800	800/1
x_1	(3)	0	1	0	0	0	1/2	600	

We solve for a new Basic Feasible Solution by constructing a new tableau in proper form using Gaussian elimination:

i. Divide the pivot row by the pivot entry;
ii. Clear to zero all the other entries in the pivot column by adding/subtracting multiples of the pivot row.

Basic Variables	Eqn no.	z	x_1	x_2	x_3	x_4	x_5	RHS
z	(0)	1	0	-2	0	0	5/2	3000
x_3	(1)	0	0	1	1/2	0	-3/4	300
x_4	(2)	0	0	1	0	1	0	800
x_1	(3)	0	1	0	0	0	1/2	600

Basic Variables	Eqn no.	z	x_1	x_2	x_3	x_4	x_5	RHS
z	(0)	1	0	-2	0	0	5/2	3000
x_2	(1)	0	0	1	1/2	0	-3/4	300
x_4	(2)	0	0	1	0	1	0	800
x_1	(3)	0	1	0	0	0	1/2	600

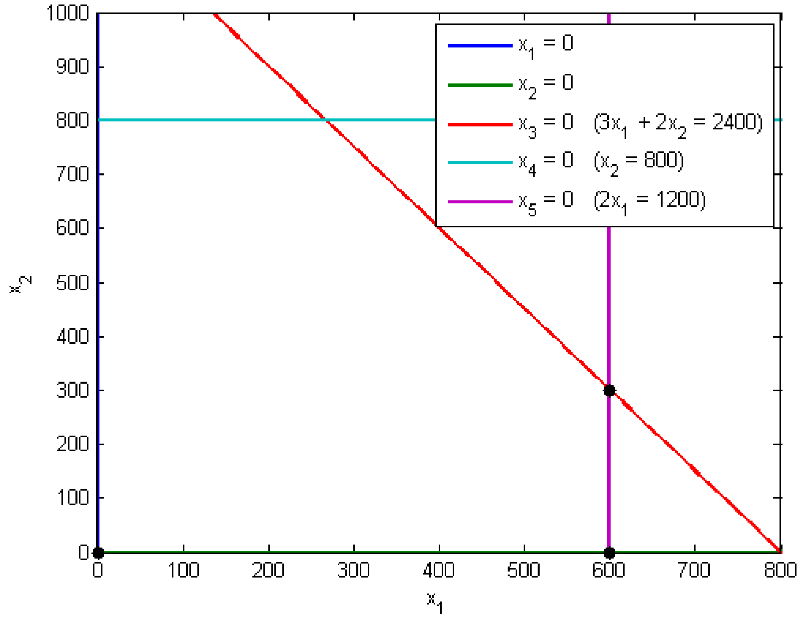
Eqn (0) gets replaced by Eqn(0) + 2*Eqn(1);
Eqn (2) gets replaced by Eqn(2) – Eqn(1);
Eqn (3) stays the same as there is already a zero in the pivot column.

TABLEAU 3:

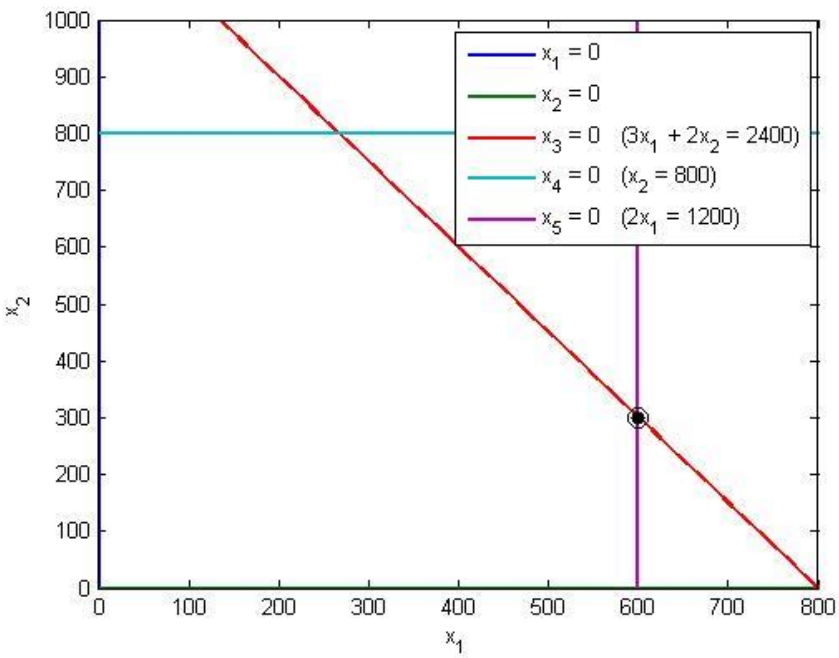
Basic Variables	Eqn no.	z	x_1	x_2	x_3	x_4	x_5	RHS
z	(0)	1	0	0	1	0	1	3600
x_2	(1)	0	0	1	1/2	0	-3/4	300
x_4	(2)	0	0	0	-1/2	1	3/4	500
x_1	(3)	0	1	0	0	0	1/2	600

Therefore the basic feasible solution associated with this tableau is

(x_1, x_2, x_3, x_4, x_5) = (600, 300, 0, 500, 0) and the value of z is 3600.



Optimality Test: The equation z + x_3 + x_5 = 3600 or z = 3600 – x_3 – x_5 says that there are no non-negative values of x_3 and x_5 that will make the objective function larger than the current value of 3600. That is to say, the coefficients of equation (0) in the tableau are all non-negative, thus this tableau is optimal and the current BFS is the optimal solution and the optimal objective value is 3600.



Note: one should check that the BFS at each stage solves the tableaux of the prior stages because each tableau is a system of linear equations that should have the same solution sets.